SIMON NEWCOMB.

SIMON NEWCOMB was born on March 12, 1835, at Wallace, Cumberland County, Nova Scotia. His parents were of New England descent temporarily settled there. He was educated by his father, and at the age of sixteen was apprenticed to a local herb doctor, but left him after two years and made his way to Maryland, in which State he spent four years teaching in various schools. An appointment on the staff of the Nautical Almanac took him to Cambridge, the headquarters of the office at that time, and he found opportunities to attend the Lawrence Scientific School, where he took the degree of B.Sc. in 1858. In 1861 he was appointed professor in the United States Navy and was detailed for work at the Naval Observatory in Washington. In 1877 he was made Superintendent of the American Ephemeris and Nautical Almanac office, and occupied this post for the next twenty years until his compulsory retirement under the regulations at the age of sixty-two. During this period he lectured for one year on political economy at Harvard, and from 1884 to 1895 was professor of mathematics and astronomy at Johns Hopkins University, and editor of the *American Journal of Mathematics*. After his retirement he continued to live and work in Washington until his death on July 11, 1909.

In 1863 he married Mary Caroline Hassler, daughter of Dr. Charles A. Hassler, U. S. Navy. His widow and three daughters survive him.

Newcomb's boyhood in a village community where life is hard and strenuous must have been full of the minor privations of life characteristic of early New England and Canadian traditions, but modified by the conditions under which he was brought up. His father was a school teacher, frequently changing from one place to another, and evidently a man of ideas more advanced than those of his generation with respect to education. Although he had to conform to current practices in the country schools where he taught, he seems to have felt that his own boy should have more mental freedom. He must have studied carefully the capacities of the young Simon and noted the direction which they took, for he made no effort to force him into work for which he was not adapted. But few books were obtainable, and those which were read were of little

intrinsic value. In his Reminiscences Newcomb says that his earliest ideas were chiefly influenced by Fowler's Phrenology and Combe's Constitution of Man. We can only judge from this that Newcomb was one of those lonely spirits that will achieve their development under any conditions, provided sufficient freedom of thought is allowed.

His somewhat erratic school education was closed at the age of sixteen by his apprenticeship to a man whose influence was of the worst possible kind for a young lad just growing into manhood. The picture which Newcomb draws of Dr. Foshay, a medical Squeers, would be notable as fiction; as a description of real life it gives one cause to wonder at the strength of character of the boy who could withstand the insidious council: "This world is all a humbug, and the biggest humbug is the best man." Fortunately, the boy had one keen desire, to learn and to know, and the unsatisfied longing finally drove him to run away from his legal master and return to his father in Maryland. He abandoned finally the idea of becoming a doctor and while in doubt as to his future career earned his living by teaching.

He evidently found time for a good deal of study in the few and second-grade books that he was able to find. Perhaps the very poverty of their treatment stimulated him to make efforts of his own, for we find him shortly afterwards sending a new proof of the binomial theorem to Professor Henry, and later answering a "crank" theorist on the Copernican doctrine in the columns of the *National Intelligencer* — his first publication. These efforts brought him to the notice of one or two scientific men, and he was not slow to avail himself of the advantages for the further pursuit of knowledge which were to be obtained through them. He discovered Bowditch's translation of Laplace and obtained a Nautical Almanac; from that time on, the main direction of his studies was never in doubt.

Although he was nominally a student of the Lawrence Scientific School for a year, and received his degree from that institution, there can be no doubt that Newcomb was, like the majority of men of his generation in America who have attained scientific fame beyond the shores of the continent, essentially self-taught. His opportunities could scarcely have been more meagre in quality or quantity, but he let none of them slip by unutilized in his desire to learn. During his four years of experience in Cambridge as a computer on the Nautical

Almanac, he came into contact with Benjamin Peirce, B. A. Gould, Runkle, and Safford, and from them he doubtless learned much, directly and indirectly; his early tendencies there received a definite scientific trend.

It was during this period that he began an investigation which resulted in the publication of one of his most important papers. At that time only a few of the minor planets had been discovered, and several theories had been put forward to account for their existence. Newcomb, collecting the various elements of these small bodies, applied to them the theory of perturbations, and showed that the distribution of their nodes and perihelia is quite arbitrary within the limits of error, and further that there was no point at which all the orbits could have intersected at any time in the past. This was generally considered as disposing of the explosion theory of Olbers. Newcomb recognized, however, that it was by no means certain that the criterion applied is sufficient. The mechanical conditions of such a swarm, under collisions and the disturbances caused by other planets, may ultimately result in an apparently arbitrary distribution. What is more interesting is the fact that even at this early date Newcomb shows in these papers that grasp of the general principles of celestial mechanics and the methods of dealing with observations, which has always been such a marked feature of all his researches. It is, in fact, a key-note to his work.

The small planets continued to interest him for several years, both in bulk and individually. We find several papers on the subject, even up to the year 1900, and he had the management of the asteroids which had been "endowed" by Watson.

His appointment as professor in the United States Navy and removal to the Observatory at Washington gave him his first opportunity to become acquainted with the practical side of astronomy. The somewhat chaotic state of the administration during the war, and for some time afterwards, again gave him the freedom to develop the subjects which interested him. Although he was making routine observations during this period, the papers which were published under his name show that his interests lay in other directions. Positional astronomy was always connected in his mind with some theory. His published papers are rarely mere sets of observations, and more generally were deductions made from the observations of others. As his knowledge gradually extended in range from the fixed

stars to the moon, he was becoming more and more impressed with the confusion creeping into exact astronomy by the different values of the constants used by different observers. He was ambitious in his programme for the future, and two plans were maturing in his mind. The first was a determination of all the constants of astronomy and their reduction to a homogeneous system. He soon saw that this would involve extended work on the theories of the planets and satellites in order to make the comparisons with observations free from empirical terms. The second problem was more special, and though not likely to demand so much time, more difficult, — the resolution of the lunar motions and the test of the law of gravitation which a comparison with the lunar theory would involve. All of his best scientific work was directed towards these two ends.

In the period between 1861 and 1877, when he became Director of the Nautical Almanac and was in a position to carry out his larger projects, his mind was active in many directions. We find short notes on a variety of subjects, — optics, finance, taxation, social science, the labor question, copyright, political economy, non-euclidean geometry. He wrote numerous reviews and popular articles, chiefly on astronomical matters. Whatever may be the ultimate fate of the views which he expressed, one feature of his writings could not fail to be valuable: he never left the reader in doubt as to his meaning. Any difficulty would be stated with that clearness and freedom from unessential detail which characterized everything he said or wrote.

By the age of thirty, and within a very few years after his arrival in Washington, Newcomb had "found himself." He had definite plans for his future, and these were more than sufficient to occupy the life-time of one man. His more important contributions to celestial mechanics were thought out before 1870, and he had a sufficiently wide knowledge of the problems of astronomy to pursue his work steadily. In fact, one sees only occasional flashes of originality after this time; he seemed to be too busy on his chosen plans to give deep thought to novelties. It is difficult to notice much difference between his earlier and his later papers, either in power or breadth of treatment. Knowledge naturally increased and he kept fully abreast of contemporary work in the astronomy of position, so that at the time of his death he was undoubtedly its chief exponent, and he may perhaps, without injustice to his predecessors, be given the foremost place amongst those who have labored for the development of this subject.

From this point on it seems proper to speak of the development of his ideas with reference to subject matter rather than in chronological order.

The position of Mars in 1862 suggested a redetermination of the solar parallax, both from the observations of that year and from all previous observations and methods. His concluded value, 8″.848, was larger than the latest determination of that much discussed constant, and in the final summary of his life-work contained in the Constants of Astronomy he abandoned it for the value 8″.790, which is quite close to the latest value 8″.806, derived by Hinks from observations of Eros. The question depends mainly on the weights to be attributed to the results obtained by different methods.

In the same line of thought is his reduction of star places in different catalogues to a homogeneous system. Each observer appears to have had his own method for reducing his observations. Any one undertaking an investigation which involved the use of different catalogues of stars was obliged to find out, first of all, how the observations had been treated. "Newcomb's aim," says Dr. Hill,* "was to eliminate as far as possible systematic errors of a personal or local nature and thus obtain a homogeneous system. This is an admirably conducted investigation and has served as a foundation for whatever has been since accomplished in this subject."

Another problem, necessary for knowledge of the constant of aberration, was a fresh determination of the velocity of light. This had been undertaken by Michelson, and though Newcomb carried through a separate investigation, partly with the help of Michelson, the results were not entirely satisfactory, and the final determination rested mainly on the former investigation. The constants of precession and nutation were also obtained. With the former appeared a catalogue of fundamental stars. The methods caused a somewhat lively discussion.

Newcomb's plans for new tables of the motions of the planets involved a recalculation of their orbits. He had already investigated those of Uranus and Neptune. What had to be done was to calculate afresh the orbits of the other planets, gather all known observations together, compare them with his preliminary tables of the planets so as to correct the constants and show any deviation not accounted for by theory, and then make final tables on a uniform system of constants. This work

* *Science*, September 17, 1909.

occupied more than twenty years. Nearly all of it was published in a series of volumes founded by Newcomb and entitled: Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac.

In starting to compute the planetary perturbations Newcomb had a choice of three methods. There was the direct method of obtaining a solution of the equations in polar coordinates, by which Laplace had treated the planetary theory; there was the method known as the variation of arbitrary constants, which had been used by Leverrier; and there was the peculiar method of Hansen. Newcomb seems to have been troubled all through his life by the difficulty of mastering complicated analysis, and always turned by preference to a simple theory which could without much trouble be prepared for computers. He thus used, at the start, Laplace's method, and we find it adopted in all his planetary theories. But he was not satisfied with Leverrier's development of the disturbing function, and some of his earliest attempts were directed towards improvements in this direction. He saw a simple method of starting by means of the eccentric anomalies, and one of his long papers contains a complete development on these lines. But he afterwards found that it did not work very well, and later he returned to the Leverrier method, with certain improvements which rendered the computations more simple.

This concluded, he was able to undertake the orbits of the four inner planets. Approximate elements only are needed in finding the perturbations, since any changes in the constants can be made easily. The elements of Leverrier served for this purpose. The investigation is published in detail in Volume III of the Astronomical Papers, and it is, at the present time, the best theory of these planets that we have. With somewhat more care it might have been made final, but knowing that his term of office was drawing to a close Newcomb seems to have been in somewhat of a hurry to finish. The calculations were executed in duplicate by computers, and the two sets of results, together with those of Leverrier, are printed side by side. In glancing over them one notices here and there considerable differences between the duplicate calculations. It would have been more satisfactory if the sources of these differences had been traced down and corrected. Duplicate calculations are indeed almost essential in such work, but they lose much of

their value if the errors revealed are not removed. There are also occasional misprints which give a feeling of insecurity to one using any special numbers for other purposes. In a later paper the secular variations of the four planets are computed by Gauss's method.

The theory was not, however, the heaviest part of the work. All known observations had to be collected and compared. It was here that Newcomb's particular genius for the organization of huge masses of material, and his firm grasp of the facts which could be deduced from them, was given free play. Over 62,000 meridian observations of Mercury, Venus, and Mars were employed for those planets alone. The work had to be done with but little addition to the ordinary force of computers employed on the Nautical Almanac. Besides the planetary comparisons, it was necessary to find the corrections to the general constants of the solar system, and also to those of the earth, such as precession and nutation, which affect every observation. All these had to be fused into a homogeneous whole so that the motions of the solar system should be made to depend on one and the same system of constants. This immense mass of work is summarized in a little volume which was published as a Supplement to the Nautical Almanac for 1897, and is entitled: Elements of the Four Inner Planets and the Fundamental Constants of Astronomy, or more briefly, Astronomical Constants. This volume gathers together Newcomb's life-work and constitutes his most enduring memorial.

The mutual perturbations of Jupiter and Saturn are so large that the problem of unravelling their motions is a much more difficult application of the planetary theory than the other six large planets require. For the successful completion of Newcomb's scheme within his life time it was necessary that these two planets should be treated by another hand and he was able to enlist the services of G. W. Hill for this purpose. Hill, indeed not only worked out the theories in full but compared them with observation and formed the tables which were the necessary complement if his labors were to be utilized to their full extent.

The publication of the tables of the planets followed rapidly. All of them bear Newcomb's name, except those of Jupiter and Saturn, for which, as we have just seen, G. W. Hill was entirely responsible. Newcomb was fortunate in securing assistance, not only from a man like Hill with a world-wide reputation, but

also from others who were far above the grade of ordinary computers, and to whom he was able to explain his methods so that they could carry the work forward with an understanding of what they were doing. Indeed, it would have been impossible for him to have covered the ground effectually without such assistance. The tables themselves contain no very notable features; a few slight improvements may be noted on the general methods for tabulating introduced by Hansen in his tables of the moon.

While carrying forward to successful completion his schemes for the planets, Newcomb continually returned to the subject which interested him above all others — the motion of the moon. As early as 1859 we find him comparing the observed places of the moon with those given in the Ephemeris, and his last paper, not yet published, deals with the same subject. Indeed, he always hoped to be able to add a set of new tables of the moon's motion to his already long list of achievements. That this did not become possible was undoubtedly a source of regret to him. But posterity will accord greater praise for what he did achieve than it would for the mere fulfilment of the ambition of a life-time.

It was not long before Newcomb's advent into the astronomical world that Hansen's tables of the moon had been introduced for the purpose of computing the moon's ephemeris in the Nautical Almanac. It was then thought that they would satisfy all the needs of astronomy. Hansen had said that they satisfied all the observations for a century back, and that the eclipses of ancient times were fairly well acounted for. Shortly before the publication of Hansen's tables, Adams had shown that the change in the mean length of the month was different from that given by Hansen, so different, indeed, that the ancient eclipses could not be reconciled if the new theoretical value were used. Newcomb, always interested in the comparison between theory and observation, wondered if Hansen's tables would agree with observations before 1750, and he started to look for data which should test the question. Early meridian observations could not be relied on, but occultations are phenomena which even the earliest scientific observers would be apt to note and which are free from many of the errors to which meridian observations are liable. At the first opportunity he went on a search over Europe for such observations. His labor was well rewarded. He was able to find sufficient ma-

terial for comparison over another century. On working it up he discovered that there was a large periodic change in the moon's longitude not indicated by theory and running through its phases in some three hundred years. The term is still unexplained.

These discoveries necessitated a more careful study of the ancient eclipses in order to see how they were affected. The new periodic term had but little influence on them; the difference between Adams's and Hansen's values for the secular change was the point at issue. In 1878 Newcomb was able to publish the results of his investigations and to give additional tables by means of which the moon would keep more nearly to its predicted place. It is true that the new terms were empirical, but that mattered little for the purposes of the Nautical Almanac. His last memoir, and one of his best, consists of a reexamination of all the known occultations since the beginning of the seventeenth century, including fresh material that he had gathered in the last twenty years. The earlier conclusions are confirmed and one or two new terms of shorter period are brought to light.

It had long been known that Hansen's theoretical investigations of the effects of the planets on the moon's motion were of doubtful accuracy. Newcomb must have had the problem of their determination, and, indeed, the whole question of Hansen's lunar theory in his mind quite early in his career. In 1868 he made a partial comparison with the only other theory approaching it in degree of accuracy, that of Delaunay. The comparison was completed in 1882 by himself and J. Meier, and it showed that the errors of the solar terms were not serious enough to affect the places of the moon when averaged over long periods of time. But Delaunay had not computed the planetary terms. In 1871 Newcomb published a method for finding them. It finally turns out to be practically a continuation of Delaunay's theory but it was far from the stage of numerical application. It was not until twenty-three years later that he continued the idea of that memoir sufficiently far to show its possibilities, though even then nothing like an extensive investigation of the whole subject was attained. Nevertheless, this second memoir is one of his ablest investigations, from the point of view of theory. In it he finds some remarkable relations that arise between the constants in the problem of four bodies. When finally developed, these rela-

tions reduce the computation of the secular accelerations to a comparatively simple problem. He himself never thought very highly of this paper, mainly because it did not fulfill the object for which it was undertaken.

His work on the subject was only completed in a long paper published a couple of years ago. Although before the appearance of Newcomb's second memoir Hill had given a practical and thoroughly effective method for computing all small perturbations, and in 1891 Radau had developed Hill's idea in considerable detail, Newcomb still clings to his own methods. He felt afterwards that this proceeding had been a mistake. In a letter to the writer he says:

"I now see had I devoted a few days or weeks to carefully studying the phases of the problem as presented not only by yourself, but Delaunay, Radau, and Hill, I might have profited immensely and been able to do my work much more easily as well as to put it into more condensed shape. But I am always repelled by intricate algebraic expressions stringing on without end without any well marked division into parts. Any attempt to manipulate them simply tires me out before I get to the end; so, however complex may be the expressions I have to deal with, I like to cut them up into parts and condense them to handle them. So I went to work trying my system without having sufficiently learned what could be done from the work of others." The whole trouble really consisted in his attempt to develop the disturbing functions by special values instead of by expansion; in spite of the help of computers such a method gives enormous calculations from which comparatively few results are obtained. The final numerical values were unfortunately erroneous in many cases, owing to a slip at the end of the work.

Almost the only excursion that Newcomb made into pure theory is contained in a paper entitled "The general integrals of planetary motion," and it consists of an attempt to show how the coordinates of a planet, under the attraction of any finite number of planets, may be represented by trigonometric series. Poincaré has used this paper as a text for his investigations into the possibilities of such developments. It is, at bottom, an extension of Delaunay's methods to any number of bodies. The various relations between the constants of integration and those present in the developments are worked out with great skill by means of Lagrange's methods for the variation of arbitrary constants. All through his theoretical work Newcomb

seems to cling to Lagrange's equations and to reach the canonical forms through them rather than to obtain the latter directly through the brief methods of Jacobi. There were, however, certain advantages in so doing. He was able to deduce, in the natural course of the work, the final canonical constants L, G, H, of Delaunay, directly from the final expressions of the rectangular coordinates in terms of the time. These are important when other perturbations of the moon, beyond those produced by the sun, have to be obtained.

A curious anomaly in the motion of Hyperion, the seventh satellite of Saturn, is satisfactorily explained in Volume III of the Astronomical Papers. Professor Hall had deduced from observation that the point of nearest approach to the planet possesses a large retrograde motion, while the ordinary theories indicate a forward motion. Newcomb traced the difficulty to an approximate commensurability between the mean motions of Hyperion and the disturbing satellite Titan. It is a case of "libration," in which an angular element oscillates instead of making complete revolutions.

In fact, Newcomb frequently had a happy faculty of hitting on the explanation of some curious anomaly and in quickly publishing a brief note on the subject. His suggestion that the difference between the observed period of the motion of the earth's pole and the period demanded by theory is due to the elasticity of the earth, has been fully confirmed by subsequent investigations. In quite another direction was the theorem that "if a fourth dimension were added to space, a closed shell could be turned inside out by flexure without stretching or tearing."

Although the paper just mentioned is perhaps Newcomb's only notable contribution to pure mathematics, he was familiar with the general directions which investigation had taken during the nineteenth century. In his presidential address before the American Mathemetical Society in 1893, he begins with a disclaimer of any right to be considered a mathematician in the modern sense of the word. But from the remarks which follow, it is evident that he had not only read but had devoted some thought to modern ideas on hyperspace, group theory, projective geometry, and the like. What is more interesting is his analysis of the manner in which the thought of the modern mathematician differs from that of the ancients, and his general conclusion that one of the chief advantages of present-day methods is their

economy in the reduction of the number of mental concepts required to express the same ideas.

Of his numerous contributions to economics I quote the statements of Professor Irving Fisher in the *British Royal Economic Journal:*

"Personally I would rank Professor Newcomb high as an economist, and am glad to find that in this judgment President Hadley heartily concurs. Doubtless Professor Newcomb leaves a large number of other admirers, yet it would seem that his economic writings did not attract the attention among economists which they deserved. . . .

"Perhaps his chief and most fruitful contribution to economic science was the distinction he showed so clearly between a 'fund' and a 'flow,' a 'fund' relating to a point of time, and a 'flow' relating to a period of time. This distinction he applied in particular to what he called 'societary circulation,' or the equation of exchange between money and goods. So far as I am aware, he was the first definitely to enunciate this equation expressing the fact that quantity of money multiplied by its velocity of circulation is equal to price level multiplied by volume of business transactions. This equation, with due amplification, represents the so-called 'quantity theory of money' in its highest form. He alone employed this same distinction between a 'fund' and a 'flow' to expose the fallacy of the 'wage-fund.' . . .

"Professor Newcomb's economic writings include many of a controversial nature. He took a vigorous part, for instance, in the discussion some years ago, as to the best standard of deferred payments, and still earlier, in the question of labor and capital. He was an advocate in general of the 'let alone' policy, though he distinguished it sharply from what he called the 'keep out' policy. In other words, he believed in free economic activity of individuals, but did not advocate the exclusion of government from economic activity.

"As to methodology, he believed that economics is a science, and consequently that the method to be followed by economic science should be the same as the method followed by science in general. He emphasized the fact that 'A law of nature can only be expressed in the form of a conditional proposition.' . . .

"In applying this idea he pointed out the fallacy of judging the effect of a tariff by the subsequent rise or fall of prices. The imposition of the tariff, as he pointed out, will make prices

higher than they *otherwise would be*, but not necessarily higher than they *had been*."

He wrote many text-books on astronomy and cognate subjects. His Popular Astronomy ran through many editions and was translated into several languages. A number of elementary text-books on mathematics were not the least of his contributions to education and they exerted considerable influence on the study of mathematics in their time. The long list of nearly four hundred publications, gathered together by Dr. Archibald in Volume XI (1905) of the *Transactions of the Royal Society of Canada*, shows titles of published addresses, reviews, numerous popular articles, several short stories, and a novel. The last he did not care to discuss, owing to some difficulties which had resulted in its publication before it was in the form in which he would have desired it to come before the public.

Education, its theory and practice, has always been a matter of interest to the people at large in the United States, and Newcomb as a professor at Johns Hopkins University and a writer of text-books naturally spoke on the subject with some authority. He was chairman of the conference on mathematics appointed by the committee on secondary school education in 1892, generally known as the Committee of Ten. The report adopted by this conference has had a large influence on the teaching of mathematics and his share in its preparation was undoubtedly considerable.

If we attempt to judge Newcomb's work, it is necessary to consider the three sides of astronomy in which he was chiefly interested. The first, purely theoretical, consists practically of mathematical investigations; the third, the purely observational side, is the function of the observatory; the second is the combination of these two, the comparison between theory and observation. There can be no doubt that Newcomb's chief title to lasting fame lies in the second of these three directions. He was a master, perhaps as great as any that the world has known, in deducing from large masses of observations the results which he needed and which would form a basis for comparison with theory. Only in his earlier years had he made any observations, and the impression one gets in reading his Reminiscences, conveys the feeling that he never cared much for that department of astronomy. On the purely mathematical side of the problems of celestial mechanics he

cannot be said to have been thoroughly at home. We have from his pen no new method for dealing with the motions of the bodies in our solar system. On the other hand, he had a thorough grasp of their mechanical relations and this nearly always served his purposes. He had too a remarkable facility for seizing on the essential point of some known method, provided it was not too complicated, improving and then adapting it so that computers could perform nearly all the numerical work. As stated above, Newcomb's predilections alway tended towards some practical result, even when he wrote a purely theoretical paper; but curiously enough, the two or three papers which represent this side of his work never reached fully successful application at his hands.

In reading Newcomb's papers one becomes impressed with his geometrical point of view of almost every subject. He seemed doubtful of analysis by means of symbols unless it were possible to get a bird's-eye view of the process and of the manner in which physical principles led up to it. Consequently, errors of theory were unusual in his publications. It has always been a matter of astonishment to the writer how easily either theoretical or numerical errors, when they did occur, could be traced down and corrected. His clear mental view of the problem was reflected in his prose, which, without graces of style, always expressed exactly what he meant.

Newcomb's influence on the thought and men of his times must be judged mainly by the condition in which he found the subjects over which his mind ranged and by his life in a land whose people were passing through the transition stage from a nation of pioneers to a firmly seated world power. Possessed of a broad mind, a wide range of knowledge, and a capacity for taking an interest in almost all matters brought to his attention, he is peculiarly well qualified to stand as a representative of the mental attitude of his generation towards the problems which confronted it. His great power of organization was applied to astronomy at a time when the subject needed just such a man to prevent confusion from falling into chaos. Deep insight was less to be desired than broad comprehension. If it cannot be said of Newcomb that he gave a new set of ideas to the world, there can be no doubt as to the value of his contributions to almost every department of his own science and to many portions of other sciences. Again, he was essentially practical in every undertaking, even to the extent of devoting

himself during his last years to a problem which he might hope to solve and in avoiding sole responsibility for any large scheme involving many years of labor. Less characteristic of his times was the impersonal attitude which he appeared to have and to expect others to have towards the objects of science. Whether this be judged as a defect or not, the task of a future historian will be rendered more simple by the absence of those qualities which would, if prominent, demand a careful investigation into the unofficial relations with the men of his time in order to obtain an accurate estimate of his influence. He seemed unable or unwilling to give that personal touch to a discussion which has played so large a part in the management of the general affairs of the nation.

Newcomb was indefatigable in his attendance at congresses, scientific meetings, and academic functions. Although he never indulged much in the lighter sides of conversation he was always ready to talk on any subject of real interest. On such occasions he would not hesitate to express any ideas which might occur to him, and sometimes mistook reticence in others for absence of ideas. He was fond of presiding over the deliberations of scientific bodies, and by his own attitude always imparted additional dignity to the proceedings. His great reputation amongst astronomers was combined with extended popular knowledge of his fame, at least on the American continent, and he received unusually numerous evidences of recognition from all parts of the world.

His naïve enjoyment of the honors which he received shows real modesty in his estimate of his own work. In correspondence he would frequently regret its defects and express the hope that others might succeed in improving it. The successful completion of a plan or an investigation seemed to call for far more praise, in his mind, than originality in conception or ability in treatment.

Newcomb's last year was spent in a courageous struggle to work in spite of much physical suffering caused by an incurable disease. As soon as he knew that time was limited for him, almost all his remaining energy was devoted to the completion of an investigation which had been in his mind for many years. It was sent to the printer towards the end of June, and he died three weeks later. Thus passed away one of the great pioneers of American science, and one who also attained a high place in the ranks of those whose merits are judged to be worthy of honor apart from time or place.

E. W. Brown.

A NEW PROOF OF WEIERSTRASS'S THEOREM CONCERNING THE FACTORIZATION OF A POWER SERIES.

BY PROFESSOR GILBERT AMES BLISS.

THE theorem which is to be proved here may be stated in the following form:

Let $f(x_1, x_2, \cdots, x_p, y)$ be a convergent series in $x_1, x_2, \cdots, x_p, y$, and such that the series $f(0, 0, \cdots, 0, y)$ begins with the term of degree n. Then $f(x_1, x_2, \cdots, x_p, y)$ is factorable in the form

$$f(x_1, x_2, \cdots, x_p, y) = (a_0 + a_1 y + a_2 y^2 + \cdots + a_{n-1} y^{n-1} + y^n)\phi(x_1, x_2, \cdots, x_p, y),$$

where $a_0, a_1, \cdots, a_{n-1}$ are convergent power series in $x_1, x_2, \cdots, x_p$ which vanish for $x_1 = x_2 = \cdots = x_p = 0$, and ϕ is a power series in $x_1, x_2, \cdots, x_p, y$ which has a constant term different from zero.

In the *Bulletin de la Société Mathématique de France* * Goursat has called attention to the fact that the proof which Weierstrass gave of this important theorem, as well as the later proofs which occur in the literature † make use of the notions of the function theory, while the theorem itself is essentially of an algebraic character. In the paper referred to he has given an elegant and elementary proof of the theorem which is in outline as follows:

By means of the substitution

$$y^n = -a_0 - a_1 y - a_2 y^2 - \cdots - a_{n-1} y^{n-1}$$

the series f can be reduced to a polynomial P of degree $n-1$ in y, whose n coefficients are convergent series in $a_0, a_1, \cdots, a_{n-1}, x_1, x_2, \cdots, x_p$. By the usual theorems in implicit function theory it is shown that the n equations found by putting these coefficients equal to zero have unique solutions for $a_0, a_1, \cdots, a_{n-1}$ as power series in $x_1, x_2, \cdots, x_p$ which vanish with $x_1, x_2,$

* "Démonstration élémentaire d'un théorème de Weierstrass," vol. 36 (1908), p. 209.

† Picard, Traité d'Analyse, vol. II, p. 243; Goursat, Cours d'Analyse, vol. II, p. 284.

$\cdots, x_p$. If the values so found are substituted in the formula

$$y^n = -a_0 - a_1 y - a_2 y^2 - \cdots - a_{n-1} y^{n-1} + \mu$$

and the series f again reduced, a polynomial P_1 of degree $n-1$ in y will be found whose coefficients are series in $x_1, x_2, \cdots, x_p, \mu$. On account of the way in which the functions $a_0, a_1, \cdots, a_{n-1}$ were determined, this polynomial P_1 has a factor μ and hence f has a factor $(a_0 + a_1 y + \cdots + a_{n-1} y^{n-1} + y^n)$.

The proof which is given below seems to the writer even more direct than that of Goursat, and it furnishes besides convenient formulas for the determination of the coefficients of the series $a_0, a_1, \cdots, a_{n-1}$.

The series f can evidently be written in the form

$$(1) \qquad f = -y^n + f_0 + f_1 y + f_2 y^2 + \cdots + f_n y^n + \cdots,$$

where the coefficients f_k are power series in $x_1, x_2, \cdots, x_p$, and $f_1, f_2, \cdots, f_n$ have no constant terms. If a convergent series

$$b = b_0 + b_1 y + b_2 y^2 + \cdots$$

having its constant term different from zero, together with n other convergent series $\mu_0, \mu_1, \cdots, \mu_{n-1}$ in $x_1, x_2, \cdots, x_p$, can be determined so that the identity

$$(2)\ b(-y^n + f_0 + f_1 y + \cdots) \equiv \mu_0 + \mu_1 y + \cdots + \mu_{n-1} y^{n-1} - y^n$$

is true, then the series ϕ of the theorem will be $1/b$ and the theorem will be proved. By comparison of the coefficients of the different powers of y in (2), the equations

$$\begin{aligned}
\mu_0 &= b_0 f_0, \\
\mu_1 &= b_0 f_1 + b_1 f_0, \\
&\cdots\cdots\cdots \\
(3)\quad \mu_{n-1} &= b_0 f_{n-1} + b_1 f_{n-2} + \cdots + b_{n-1} f_0, \\
b_0 - 1 &= b_0 f_n + b_1 f_{n-1} + \cdots + b_{n-1} f_1 + b_n f_0, \\
&\cdots\cdots\cdots \\
b_k &= b_0 f_{n+k} + b_1 f_{n+k-1} + \cdots + b_{k-1} f_{n+1} + b_k f_n + \cdots + b_{n+k} f_0, \\
&\cdots\cdots\cdots
\end{aligned}$$

are found. These equations determine uniquely the coefficients of the series $b_0, b_1, \cdots$; $\mu_0, \mu_1, \cdots, \mu_{n-1}$ as rational integral functions with positive coefficients of the coefficients of the series $f_0, f_1, f_2, \cdots$. For on account of the fact that $f_0, f_1, \cdots, f_n$ have no constant terms, the terms of order m in b_k can be determined from the last equation in the form just described as soon as the terms of order m and less in $b_0, b_1, \cdots, b_{k-1}$, and those of order $m-1$ and less in $b_k, b_{k+1}, \cdots, b_{k+n}$, are known. Suppose, for example, that the terms of order zero of all the b's up to and including b_{k+n} have been computed. From them the terms of order one of $b_0, b_1, \cdots, b_{k+(n-1)n}$ can be found; then the terms of order two of $b_0, b_1, \cdots, b_{k+(n-2)n}$; and so on, until the terms of order m of $b_0, b_1, \cdots, b_k$ are obtained. Hence step by step the terms of the different orders can be determined for all of the series b_k, and hence for all the series μ. It is evident, therefore, that *if there exist convergent series* $b, \mu_0, \mu_1, \cdots, \mu_{n-1}$ *satisfying identically the relation* (2), *then those series have coefficients which are uniquely determined by the relations* (3). *Furthermore if a function* F *of the form* (1) *can be found for which the coefficients in* $F_0, F_1, \cdots$ *are positive and greater in numerical value, respectively, than those of* f, *and such that the corresponding series* $B, M_0, M_1, \cdots, M_{n-1}$ *for* F *are convergent, then the series* $b, \mu_0, \mu_1, \cdots, \mu_{n-1}$ *for* f *will also be convergent.*

A function F of the type desired can readily be found. The series f can be supposed without loss of generality to be convergent for $x_1 = x_2 = \cdots = x_p = y = 1$. For if it were convergent for $|x_i| \leqq \rho_i$, $|y| \leqq \rho$, it would only be necessary to make the transformation $x_i = \rho_i x_i'$, $y = \rho y'$ in order to have a series with the desired property. If the values $x_1 = x_2 = \cdots = x_p = y = 1$ are substituted in f, the resulting series is the series of the coefficients of f and is absolutely convergent. Hence each coefficient of f is in absolute value less than a certain positive constant N. For the function F, then, let

$$F_0 = F_1 = \cdots = F_n = N\left[\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_p)} - 1\right],$$

$$F_{n+k} = N\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_p)},$$

where $k = 1, 2, \cdots, \infty$. Every coefficient of F is positive and greater in absolute value than the corresponding coefficient of f.

For the function F the relation (2) after a simple transformation takes the form

$$\frac{B}{1-y}\left[-y^n+\frac{N}{(1-x_1)(1-x_2)\cdots(1-x_p)}-N\right.$$
$$\left.+(1+N)y^{n+1}\right]\equiv M_0+M_1y+\cdots+M_{n-1}y^{n-1}-y^n.$$

Series B, M_0, M_1, $\cdots$, M_{n-1} of the type desired can be found satisfying this identity provided that series C, M_0, M_1, $\cdots$, M_{n-1} of similar type can be found satisfying the identity

$$-y^n+F_0+cy^{n+1}\equiv(M_0+M_1y+\cdots+M_{n-1}y^{n-1}-y^n)C,$$

where $c=1+N$. The latter, however, can be satisfied by a linear function $C=M-cy$. For by comparing the coefficients of y^n it is found that $M=1-cM_{n-1}$, and by comparing the coefficients of the lower powers

$$\begin{aligned}
M_0 &= cM_0M_{n-1}+F_0,\\
-cM_0+M_1 &= cM_1M_{n-1},\\
-cM_1+M_2 &= cM_2M_{n-1},\\
&\cdots\cdots\cdots\cdots\\
-cM_{n-2}+M_{n-1} &= cM_{n-1}M_{n-1}.
\end{aligned}$$

From the well known theorems concerning implicit functions it follows that these equations have solutions M_0, M_1, $\cdots$, M_{n-1} which are power series in x_1, x_2, $\cdots$, x_p, vanishing when $x_1=x_2=\cdots=x_p=0$. Hence the theorem is proved.

It should be noted that the dominant function F and the implicit function theory have been used only in the proof of the convergence of the series b and μ_1, μ_2, $\cdots$, μ_{n-1}. The computation of the coefficients of these series is effected by means of the equations (3).

THE UNIVERSITY OF CHICAGO,
November, 1909.

ON SOME THEOREMS IN THE LIE THEORY.

BY PROFESSOR L. D. AMES.

(Read before the Southwestern Section of the American Mathematical Society, November 27, 1909.)

THE well-known treatment of the infinitesimal transformation in the Lie theory makes use of power series, assuming that the functions involved are analytic. The treatment here given demands at most the existence of second partial derivatives. Since no use is made of the group theory, the proofs here given may be used either in connection with the group theory or apart from it.

Three theorems are stated in this paper, of which Theorem I. was proved in a previous paper.* Of the three, any one is an immediate result of the other two taken together. They may be compactly stated as follows:

Given any differential equation of the form

$$\Omega(x, y, y') = 0 \tag{1}$$

which, in a given region R, can be written in the form linear in y'

$$\Omega_1 \equiv X(x, y)y' - Y(x, y) = 0, \tag{2}$$

and of which $\omega(x, y) = c$ is an integral; let $\xi(x, y)$ and $\eta(x, y)$ be such functions that

$$X\eta - Y\xi \not\equiv 0.$$

Consider the statements

A. *The equation*

$$\frac{X}{X\eta - Y\xi}y' - \frac{Y}{X\eta - Y\xi} = 0 \tag{3}$$

is exact. That is, $1/(X\eta - Y\xi)$ is an integrating factor of (2).

B.

$$\xi\frac{\partial\Omega}{\partial x} + \eta\frac{\partial\Omega}{\partial y} + \eta'\frac{\partial\Omega}{\partial y'} = 0,\dagger \tag{4}$$

* BULLETIN, vol. 15, no. 8 (May, 1909).

† In the Lie theory this is the condition for invariance under the extended transformation

$$Uf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}.$$

where

$$\eta' = \frac{d\eta}{dx} - y' \frac{d\xi}{dx}.$$

C.

$$\xi \frac{\partial \omega}{\partial x} + \eta \frac{\partial \omega}{\partial y} = F(\omega).^* \tag{5}$$

THEOREM I. B *is a necessary and sufficient condition for* A.

THEOREM II. C *is a necessary and sufficient condition for* B.

THEOREM III. C *is a necessary and sufficient condition for* A.

The proofs are in outline as follows:

Theorem I. The familiar condition that (3) be exact, after obvious simplifications, is (4).

Theorem II. Differentiate (5) totally along $\omega = c$. The result, after obvious substitutions, is (4).

Theorem III. Assume that M is the unknown integrating factor of (2). Writing the exact equation in two forms, we obtain an equation which we can solve for M. This proof does not assume a previous knowledge of the form of the integrating factor.

More detailed proofs of Theorems II. and III. follow.

Proof of Theorem II. First to prove C sufficient. Since $\omega = c$ is an integral of (2) we may write (2) in the form

$$\Omega_1 \equiv M \frac{\partial \omega}{\partial x} + My' \frac{\partial \omega}{\partial y} = 0. \tag{6}$$

As in the previous paper, the partial derivatives of Ω with respect to x, y and y' are proportional to those of Ω_1. Form these and substitute in the equation obtained by differentiating (5) totally along the curve $\omega = c$. The result is (4).

To prove C necessary we assume (4), and retrace the steps of

* In the Lie theory this is the condition that the family of integral curves $\omega(x, y) = c$ is invariant under the transformation

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}.$$

Throughout this paper $F(\omega)$ is to be read "some function of ω" when (5) is a necessary condition, and "any function of ω" when (5) is a sufficient condition.

the above proof, obtaining

$$\frac{d}{dx}\left[\xi\frac{\partial\omega}{\partial x}+\eta\frac{\partial\omega}{\partial y}\right]=0. \tag{7}$$

Noting that y and y' are functions of x and c, the integration of (7) gives

$$\xi\frac{\partial\omega}{\partial x}+\eta\frac{\partial\omega}{\partial y}=F(c)=F(\omega). \tag{5}$$

Proof of Theorem III. First to prove C sufficient. Equation (2) has an integrating factor which we will call M. Then the equation

$$MXy'-MY=0$$

is exact. It is therefore of the form

$$\frac{\partial\omega_1}{\partial y}y'+\frac{\partial\omega_1}{\partial x}=0,$$

where ω is some function of ω_1, $\omega=\phi(\omega_1)$. Hence

$$\frac{\partial\omega}{\partial x}=\phi'(\omega_1)\frac{\partial\omega_1}{\partial x}=-\phi'(\omega_1)MY,$$

$$\frac{\partial\omega}{\partial y}=\phi'(\omega_1)\frac{\partial\omega_1}{\partial y}=\phi'(\omega_1)MX.$$

Putting these values in (5) and solving for M, we obtain

$$M=\psi(\omega_1)/(X\eta-Y\xi),$$

where ψ is some function of ω_1. Hence $1/(X\eta-Y\xi)$ is also an integrating factor.

To prove that (5) is necessary, let $1/(X\eta-Y\xi)$ be a particular integrating factor, and $\omega_1(x,y)=c$ the corresponding integral. Let $\omega(x,y)=c$ be any integral. Then the corresponding integrating factor is $\chi(\omega_1)/(X\eta-Y\xi)$, where χ is some function of ω_1. Hence the equations

$$\frac{X\chi}{X\eta-Y\xi}y'-\frac{Y\chi}{X\eta-Y\xi}=0$$

and

$$\frac{\partial \omega}{\partial y} y' + \frac{\partial \omega}{\partial x} = 0$$

are identical. Hence

$$\xi \frac{\partial \omega}{\partial x} + \eta \frac{\partial \omega}{\partial y} = \chi(\omega_1) = F(\omega). \tag{5}$$

UNIVERSITY OF MISSOURI,
November, 1909.

ON THE DISCONTINUOUS ζ-GROUPS DEFINED BY RATIONAL NORMAL CURVES IN A SPACE OF n DIMENSIONS.

BY PROFESSOR J. W. YOUNG.

(Read before the Chicago Section of the American Mathematical Society, January 1, 1910.)

THE present note completes in an important particular a paper* which I presented to the Society some two years ago. I there considered the discontinuous groups Γ_n of linear fractional transformations

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \tag{1}$$

on the complex variable ζ, defined as follows by a rational normal curve C_n in a space S_n of n dimensions: The given C_n is transformed into itself by a group of ∞^3 collineations

$$z_i' = \sum_{k=0}^{n} a_{ik} z_k \qquad (i = 0, 1, \cdots, n) \tag{2}$$

in S_n. Each of these collineations subjects the parameter ζ of the points of C_n to a substitution (1), so that the continuous three-parameter groups of transformations (1) and (2) are simply isomorphic. If now the transformations (2) be restricted to those whose coefficients a_{ik} are rational integers with determinant $|a_{ik}| = 1$, the resulting subgroup of the three-parameter group of transformations (2) will be properly discontinuous.

* "A fundamental invariant of the discontinuous ζ-groups defined by the normal curves of order n in a space of n dimensions," BULLETIN, vol. 14 (1908), pp. 363–367.

The same is then true of the group of transformations (1) which corresponds in the isomorphism already noted to the discontinuous subgroup just defined.

In the paper cited above I proved the following theorem:

In the discontinuous ζ-groups defined as indicated by any normal curve C_n in a space S_n the period ω of any elliptic substitution must satisfy a relation of the form

$$\sin\frac{(n+1)\pi}{\omega} = J\sin\frac{\pi}{\omega},$$

where J is an integer (positive, negative, or zero).

This theorem gives a necessary condition on the positive integers ω that may be periods of elliptic substitutions in any Γ_n. In particular, the numbers n, $n+1$, $n+2$ are all possible values of ω under this condition. It has not been shown hitherto, however, that groups Γ_n really exist containing substitutions of these periods. In the present note it is shown how to construct a C_n which will define in the manner indicated a group Γ_n containing any given substitution of period $n+1$ or $n+2$. The solution $\omega = n$ is not in general possible when $n > 2$.

We recall first the necessary and sufficient condition that a collineation in S_n leave invariant a normal C_n.* We will suppose that the collineation produces on the C_n a projectivity with two distinct double points, P_0 and P_n. The osculating k-spaces ($k = 1, \cdots, n-1$) at P_0 and P_n are then invariant; and the osculating k-space at P_0 determines with the osculating $(n-k)$-space at P_n a double point P_k of the collineation.

The invariant configuration of the collineation is then a complete $(n+1)$-point in S_n,† two of whose vertices are on the curve and which is completely determined by these two points and the curve. An $(n+1)$-point thus situated with reference to a C_n will be said to *osculate* the C_n at the two points. If this invariant $(n+1)$-point is taken as the fundamental $(n+1)$-point of the coordinate system, the points $(0, 0, \cdots, 1)$ and $(1, 0, \cdots, 0)$ being taken as P_0 and P_n respectively, the equations of the C_n are of the form

*Gino Loria, "Sulle curve razionali normali in un spazio an n dimensioni," *Giornale di Matematiche*, vol. 26 (1888), p. 345.

†It may, of course, happen that more than $n+1$ points are left invariant by the collineation. But in any case the invariant configuration must include a complete $(n+1)$-point of the specified kind.

$$z_i = c_i \zeta_i^{n-i} \qquad (i = 0, 1, \cdots, n), \tag{3}$$

where the c_i are any constants.

It now follows readily that the roots of the characteristic equation of any collineation leaving the C_n invariant must be proportional to the powers $\mu^n, \mu^{n-1}, \cdots, \mu, 1$ of a number μ. But this condition with the preceding is at once seen to be also sufficient. We have then

THEOREM 1. *The necessary and sufficient condition that a collineation in S_n (with at least two distinct double points *) leave a rational normal curve C_n invariant is that its invariant configuration contain a complete $(n+1)$-point and that the roots λ of its characteristic equation be proportional to the powers $\mu^n, \mu^{n-1}, \cdots, \mu, 1$ of a number μ. A collineation satisfying this condition leaves every normal curve invariant which is osculated by the invariant $(n+1)$-point at the points $\lambda = \rho\mu^n$ and $\lambda = \rho$.*

We are now in a position to write down the equations of any C_n which is left invariant by a given collineation, if such C_n's exist. Let π be any collineation satisfying the condition of Theorem 1. Let

$$Z_0 = 0, \quad Z_1 = 0, \quad \cdots, \quad Z_n = 0$$

be the equations the $n+1$ invariant $(n-1)$-spaces of the invariant $(n+1)$-point of π. The collineation may then be written

$$\rho Z_i' = \mu^{n-i} Z_i \qquad (i = 0, 1, \cdots, n).$$

Any C_n left invariant by π is then given by the equations

$$Z_i = c_i \zeta^{n-i} \qquad (i = 0, 1, \cdots, n). \tag{4}$$

To complete the proof of the existence theorem which we have in view it is necessary only to show that collineations of periods $n+2$ and $n+1$ with integral coefficients and determinant 1 exist in S_n which satisfy the conditions of Theorem 1. Representing a collineation by the matrix of its coefficients we find readily that the two collineations

* It is readily seen that, if the collineation has two distinct double points, it must have at least $n+1$, two of which are on the curve. Collineations with only a single double point and leaving a C_n invariant exist, moreover; a fact which was overlooked by Loria in the paper quoted above. Cf. in this connection the latter part of the present paper.

$$\pi_1 = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

and

$$\pi_2 = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{vmatrix}$$

are respectively of periods $n+2$ and $n+1$. Indeed their characteristic equations are respectively

for π_1: $\lambda^{n+1} + \lambda^n + \cdots + \lambda + 1 = 0$;

for π_2: $\lambda^{n+1} - 1 = 0$.

The collineations π_1, π_2 are then readily seen to satisfy the conditions of Theorem 1. This completes the proof of

THEOREM 2. *Rational normal curves C_n exist which define in the way indicated discontinuous ζ-groups containing elliptic substitutions of period $n+1$ or period $n+2$.*

This result is of considerable interest on account of the fact that it proves for the first time that the class of ζ-groups Γ_n ($n > 2$) is indeed more extensive than the class of groups Γ_2. In particular the classes of groups Γ_3 and Γ_4 contain groups with elliptic substitutions of period 5; no extensive class of such groups have as yet been defined arithmetically. The detailed discussion of the groups defined by normal curves C_3 and C_4 is, therefore, greatly to be desired. This discussion doubtless would lead to results of much interest in themselves, and might point the way for a more detailed treatment of the general case, which at present seems to be of too great complexity for immediate treatment. In this connection it may be noted that the curves C_n defined by equations (4) will lead to groups in which the

elliptic substitutions of periods $n+1$ or $n+2$ have double points at $\zeta=0$ and $\zeta=\infty$. Their coefficients will therefore be imaginary. If, however, the curve C_n is written

$$Z_i = c_i(\zeta-\eta)^{n-i}(\zeta-\bar{\eta})^i \quad (i=0, 1, \cdots, n),$$

these double points will be at $\zeta=\eta$, $\zeta=\bar{\eta}$ respectively. It is, therefore, possible to write down at once the equations of a C_n for which the corresponding Γ_n contains any given elliptic substitution of period $n+1$ or $n+2$ of form (1).

It may also be noted that *the period $\omega=n$, which satisfies the necessary condition of the earlier paper, is not in general possible if $n>2$.* For $n=3$, for example, there exist no collineations in S_3 of period 3 which satisfy the conditions of Theorem 1.

Finally we enquire regarding the existence of parabolic substitutions in a Γ_n. Supposing the coordinate system so chosen as to render the equations of the C_n of the form

$$z_i = \zeta^{n-i} \qquad (i=0, 1, \cdots, n), \tag{5}$$

any parabolic substitution may be taken to be

$$\zeta' = \zeta + 1.$$

The corresponding collineation leaving c_n invariant is then readily calculated from the equations

$$z_i' = (\zeta+1)^{n-i} \qquad (i=0, 1, \cdots, n),$$

and substituting from (5). The characteristic equation of this collineation is found to be

$$(\lambda-1)^{n+1}=0.$$

This shows first that *a collineation in S_n which produces on an invariant C_n a parabolic projectivity has only a single double point.**

It shows, furthermore, that any C_n defining a Γ_n with parabolic substitutions must contain a point with integral coordinates. This is a generalization of the corresponding fact for the case $n=2$. Fricke† used this in the case $n=2$ to show that every Γ_2 with parabolic substitutions must be commensurable with the elliptic modular group. His argument, however, does

* Cf. in this connection the footnote on p. 365.

† Fricke-Klein, Automorphe Functionen, vol. 1, p. 518.

not seem to apply when $n>2$. This suggests the important question: Can any fundamental circle group (discontinuous ζ-group that is representable with real coefficients) be defined arithmetically as a Γ_n for a sufficiently high value of n?

UNIVERSITY OF ILLINOIS,
December, 1909.

A NEW ANALYTICAL EXPRESSION FOR THE NUMBER π, AND SOME HISTORICAL CONSIDERATIONS.

BY DR. G. VACCA.

IN the common exposition of the history of mathematics, more attention is given to results than to methods, and it is only rarely that old theorems and demonstrations are translated into the modern and living mathematical language. The great mathematicians of the eighteenth century (Euler, Lagrange, and others) have made important contributions to this difficult work. It seems to me that even to-day something can be done in this direction, and many new results can be obtained by a careful reading of the ancient classics of mathematics. The following is an example:

Let $f(x)$ be a function defined by the relation

$$f(x)=\tfrac{1}{2}(x+|x|)$$

and suppose that $f^n(x)$ means the result of the operation f applied n times to the number x. Then we have, if $i=\sqrt{-1}$,

$$\frac{2}{\pi}=\lim_{n=\infty} f^n(i). \tag{1}$$

This elegant theorem can be easily proved. It gives a possible analytical definition of the number π.

It is now to be observed that this *new* formula is only the analytical expression (using the geometrical representation of complex quantities of Gauss) of a series of points (approaching the vertex) of the quadratrix (*τετραγωνίζουσα*) of Deinostratos.*

We may try to transform the second member of (1) into a real expression. This can be done by elementary methods. But the result is nothing else than the well-known infinite product of complicated quadratic radicals first given by Vieta.†

* Pappi Coll. Math., lib. IV, prop. 25.

† Vieta, Opera, ed. Schooten, Lugd. Batav., 1646, p. 400.

But with the use of trigonometric functions the reduction of the formula (1) is immediate. We find

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cdots,$$

an elegant expression given by Euler.*

Note. We can avoid in the formula (1) the use of the intermediate symbol f by employing the notations followed by the Formulaire of Peano.

First, the parentheses in the expression $f(x)$ are useless, and were never employed by classic writers (Lagrange, Gauss).

Secondly, given an expression A containing the variable x, it is useful to denote by $A|x$ the operator or functional symbol f such that $fx = A$. The operation $|$ called *inversion*, was first suggested by Eisenstein.†

Thirdly, instead of the modern symbol $|x|$ for the modulus of the quantity x, it is more convenient to use the classical symbol of Cauchy ‡ mod x, where mod is an operator prefixed to the variable, like sin, cos, $\cdots$.

Lastly, the ordinary symbol lim has some indeterminacy: it does not express in what field the variable should vary while tending to ∞. A complete expression § to indicate the limiting value of the function f when the variable, varying in the field u, tends to the value a should be of the form

$$\lim (f, u, a).$$

Then, if N denotes the whole class of the positive integers, we have

$$\frac{2}{\pi} = \lim\left\{\left[\left(\frac{x + \operatorname{mod} x}{2}\,\middle|\, x\right)^{n} i\right] n, N, \infty\right\},$$

and this formula is complete in itself, and denotes a truth of the same kind as the formula $7 + 5 = 12$.

GENOA, ITALY,
December, 1909.

* *Comm. Petropol.*, vol. 9 (1737).

† Eisenstein, Math. Abhandl., Berlin, 1847, pp. 71, 91.

‡ Cauchy, Exercices, vol. 4 (1829), p. 47; Oeuvres ser. 2, vol. 9, p. 95.

§ Cf. G. Peano "Sur la définition de la limite d'une fonction," *American Journal of Mathematics*, vol. 17 (1894).

HERMITE'S WORKS.

Oeuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences par EMILE PICARD. Vol. II. Paris, Gauthier-Villars, 1908. 8vo. 520 pp.

IN a former number of the BULLETIN * we have given a biographical sketch of Hermite and endeavored to set forth the commanding position he occupied among French mathematicians during the last century. We shall therefore give only a brief notice of the present volume. Its 37 papers bring his publication down to the year 1872 and their chief interest centers about the elliptic functions, the solution of algebraic equations, and the theory of numbers. The reader will perhaps best obtain an idea of the rich contents of this volume if we number its papers in the order in which they occur, arrange them in groups, and indicate briefly their nature. A few of the more important papers will however be analysed at length.

Elementary Algebra.

28) A note of 28 pages written for Gerono and Roguet's Cours, edition of 1856, on homogeneous forms of the second degree in n variables.

Differential Calculus.

37) Elimination of arbitrary functions. Extracted from Hermite's Cours at the Ecole Polytechnique, 1873.

29) Short proof of $1/\rho = d\phi/ds$.

Integral Calculus.

30), 33), 34), 36) Short notes on some integrals, such as

$$\int \frac{x^m dx}{\sqrt{1-x^2}}.$$

Theory of Substitution Groups.

19) Discussion of the nature of $\theta(i)$ in order that

$$\begin{bmatrix} z_i \\ z_{\theta(i)} \end{bmatrix}$$

* 2d ser., vol. 13, p. 182.

may represent a substitution of $z_0, z_1 \cdots, z_{n-1}$. Application to the group of order 168 on 7 letters.

Invariants.

1), 20) Short notes on cubic ternary forms.
9) Reduction of the binary cubic.
10) Resultant of three ternary quadratic forms.
26) The invariants of the binary quintic.
27) The skew invariant of the binary sextic.

Theory of Numbers.

8) A modification of Dirichlet's method of determining the number of classes of binary quadratic forms having a given determinant.

Theory of Functions.

8) Determination of a polynomial $P(x)$ of degree m such that the sum of the squares of the differences between $P(x)$ and a given function for n values of x, $n \geq m$, shall, when each difference is multiplied by a constant, be a minimum. A different solution from that given by Tchebychef.

21), 22), 23), 24) These four papers occupy 54 pages and constitute one of the most interesting features of this volume. They are a study of different functions of more than one variable which have properties in common with Legendre's function X_n in one variable.

Hermite begins by observing the fundamental nature of the θ's in the elliptic and abelian functions. These θ's are sums of terms of the type $e^{-\phi(x+h, y+h_1, \cdots)}$, ϕ being a quadratic form. The development of this term gives the series

$$\sum \frac{h^n h_1^{n'} \cdots}{n!\, n'! \cdots} U_{n, n' \cdots},$$

where the U's are polynomials in $x, y \cdots$ of degree $n + n' + \cdots$. These U's and a set of associate polynomials V are the first functions which Hermite considered. Their importance in analysis is due, so Hermite thought, partly to the fact that they have many properties of the X_n function and partly to their intimate relation with the abelian functions. After some intermediate steps Hermite arrives at another set of functions which

seem to please him better. Let us restrict ourselves to two variables, and set

$$L(a,\ b) = 1 - 2ax - 2by + a^2 + b^2,$$

$$Q(a,\ b) = 1 - 2ax - 2by + a^2(1 - y^2) + 2abxy + b^2(1 - x^2).$$

On developing we get

$$\frac{1}{Q} = \sum a^m b^n U_{m,\,n}, \qquad \frac{1}{L} = \sum a^m b^n V_{m,\,n},$$

where $U_{m,\,n}$, $V_{m,\,n}$ are polynomials in x, y of degree $m + n$. The analogy between $U_{m,\,n}$ and X_n is most perfect. We note that the integral relations analogous to

$$\int_{-1}^{1} X_m X_n dx = 0, \quad \text{etc.}$$

depend on the evaluation of integrals such as

$$\iint \frac{dxdy}{L(ab)L(a'b')} = \frac{\pi}{ab' - ba'} \arctan \frac{ab' - ba'}{1 - aa' - bb'},$$

$$\iint \frac{dxdy}{L(ab)Q(a'b')} = \frac{\pi}{aa' + bb'} \log \frac{1}{1 - aa' - bb'},$$

the field of integration being the interior of the unit circle $x^2 + y^2 \leq 1$. We recommend the evaluation of these integrals as an excellent exercise of the reader's ingenuity.

Elementary Elliptic Functions.

13) An elementary presentation of this theory occupying 114 pages. It is taken from a note in Lacroix's Calculus, edition of 1862. This little gem will always have a historical value as coming from the hand of a great master of this theory ; but even to-day one finds points of view which are suggestive and worthy of thought.

14) A transformation of the third order by means of invariants.

17) A method for rapidly computing the coefficients of x^n in the development of sn, en, dn.

18) Development of $\sqrt[4]{k}$, $\sqrt[4]{k'}$ in series involving q.

31) Development of elliptic integrals of the first and second species.

32) Development of k^2 in powers of $(\omega - i)/(\omega + i)$.

33) Geometric demonstration of the addition theorem.

Application of Elliptic Functions to the Solution of Equations.

2) This is the world-famous solution of the quintic. For three centuries mathematicians had tried in vain to effect its solution.

3) A delightful letter written by the aged Hermite in 1900 to Tannery, giving a proof of the formulas relating to the linear transformation of the modular functions

$$\phi(\omega) = \sqrt[4]{k}, \quad \psi(\omega) = \sqrt[4]{k'}$$

employed in 2). How touching is the close of this letter. "De ma proximitée de l'Espagne je rapporte des cigarettes d'Espagnoles; si vous ne venez pas (after his return to Paris) en fumer avec votre collaborateur d'aujourd'hui, votre professeur d'autrefois, c'est que vous avez le coeur d'un tigre. Totus tuus et toto corde."

4), 5) Application of the invariant theory and the elliptic functions to the solution of the biquadratic. Here is also introduced Hermite's χ function with a table of its linear transformations.

6) Elliptic modular equations. This is a famous memoir, published serially in the *Comptes Rendus* for 1859 and then as a monograph. The point of departure of Hermite's researches on this subject is the actual determination of the resolvents of 7th and 11th degrees of the modular equations $M = 0$ of 8th and 12th degrees, whose existence Galois first pointed out in 1832. The coefficients of all the terms except the absolute term may be found without excessive labor; the last term turns out to be the square root of the discriminant $\sqrt{D}$ of $M = 0$. The calculation of D for equations of so high degree by general methods would be utterly impracticable; it is therefore necessary to make use of the peculiar properties of $M = 0$. Doing this, it is easy to show that D has the form for a prime degree n

$$D = u^{n+1}(1 - u^8)^{n+\epsilon} P^2(u),$$

where u is the function $\phi(\omega)$ above, ϵ the Legendrian symbol $\left(\frac{2}{n}\right)$ and P a polynomial with distinct roots of degree

$$\nu = \frac{n^2 - 1}{8} - \frac{n + \epsilon}{2}$$

in u^8. The determination of the coefficients of P requires now a stroke of genius. Hermite observes that the roots of $P(u)$ correspond to values of u or, what is the same thing, of ω for which $M=0$ has equal roots. His transformation theory of $\phi(\omega)$ shows him at once that this requires that ω shall satisfy the quadratic equation

$$P\omega^2 + 2Q\omega + R = 0 \tag{1}$$

with negative determinant $-\Delta$ of the type

$$\Delta = (8\delta - 3n)(n - 2\delta) \text{ or } \Delta = 8\delta(n - 8\delta), \tag{2}$$

where δ is any integer such that $\Delta > 0$. Thus for $n=7$ and 11, and these are the two cases Hermite is especially interested in as we observed above, we have: $n=7$, only one determinant, viz., $\Delta = 3$; $n=11$, only two, viz., $\Delta = 7$ and $\Delta = 24$. But conversely, with certain easily determined exceptions, for each of the classes of binary forms corresponding to the determinants (2) there exist two or six quadratic forms (P, Q, R) whose coefficients in (1) give values of ω for which two of the roots of $M=0$ are equal. Thus with hardly any calculation Herm te finds that for

$$n=7, \quad P = 1 - u^8 + u^{16}.$$

The case $n=11$ is more difficult. To determine P in this case Hermite observes that the values of ω given by (1) correspond to complex multiplication.* The resulting modular equations must therefore split up into rational factors. Applying this fact to the modular equation of 12th degree which Sohnke had calculated in 1837, Hermite was able to compute P for this case with relatively little labor.

7) Resolvent of the 7th degree of the modular equation ot the 8th degree.

11) Employs the skew invariant of the quintic to find a quintic resolvent of the general equation of 5th degree.

25) A long memoir of 87 pages devoted to a study of the quintic. The invariant theory is freely used. An especially interesting result is a set of criteria for the reality of the roots.

* For a glimpse of this fascinating subject, all too little known in this country, the reader is referred to a review by the author in the BULLETIN, 2d ser., vol. 6 (1900), p. 460.

Application of Elliptic Functions to Quadratic Forms.

12), 15) To make clear to the reader the import of these two papers, in which Hermite's genius shines with such lustre, it will be necessary to go back a little. Abel stated without proof that the equation of transformation of the elliptic functions could be solved by radicals in the case of complex multiplication. This precious fact lay long buried and forgotten until Kronecker resurrected it and made it the point of departure of a long series of brilliant discoveries. He found that the equations in these singular cases split up into rational factors which stand in the most intimate relation with the number of classes of quadratic forms with negative determinants. By this means he found eight fundamental relations which being published (1857–1861) with scarcely any proof were long the wonder and admiration (or envy) of his fellow workers in this field. As an illustration let us cite one of them.

$$F(2m) + 2F(2m - 1^2) + 2F(2m - 2^2) + \cdots = 2\Phi(m),$$

where $F(m)$ is the number of classes of determinant $-m$ in which at least one of the outer coefficients is odd and $\Phi(m)$ is the sum of the divisors of m.

Right well might the mathematical world be astonished when Hermite showed how some at least of them might be obtained by elementary means, using a method already employed by Jacobi. Before explaining how Hermite did this, let us illustrate the method by a simple example which the reader can follow in detail. We know that

$$\frac{4k^2}{\pi^2} = (1 + 2q + 2q^4 + 2q^9 + \cdots)^4 \tag{3}$$

and also that this is equal to

$$1 + 8\sum_n \frac{nq^n}{1 + (-1)^n q^n} = 1 + 8\sum_{m,n} n(-)^{m(n+1)} q^{(m+1)n}. \tag{4}$$

Now the exponent of q in the last member of (4) can take any integral value ν, whereas when we raise the parenthesis to the fourth power in (3) it has the form

$$n_1^4 + n_2^4 + n_3^4 + n_4^4.$$

This gives at once the celebrated theorem of Fermat; every

integer can be represented as the sum of four squares. If we consider the coefficient of q^ν in (4) more carefully, we observe that it is eight times the sum of the divisors of ν, if ν is odd; and 24 times the sum of the odd divisors of ν, if ν is even. Thus we get with no difficulty a celebrated theorem due to Eisenstein: the number of representations of the integer ν is eight times the sum of its divisors, when ν is odd; and 24 times the sum of its odd divisor when ν is even. This is the theorem whose proof as given by Eisenstein requires an elaborate knowledge of quaternary quadratic forms.* Let us now consider one of the cases considered by Hermite. Letting $\Theta(z)$, $H(z)$, $\Theta_1(z)$ and $H_1(z)$ be the functions of Jacobi, setting $z = 2Kx/\pi$,

$$\Theta(z) = 1 - 2q\cos 2x + \cdots, \quad \Theta_1(z) = 1 + 2q\cos 2x + \cdots,$$

$$H(z) = 2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + \cdots,$$

$$H_1(z) = 2\sqrt[4]{q}\cos x + 2\sqrt[4]{q^9}\cos 3x + \cdots,$$

Hermite finds

$$(5)\quad \frac{K}{2\pi}\sqrt{\frac{2kKH^2(z)\Theta_1(z)}{\pi\qquad\Theta^2(z)}} = A\Theta_1(z) - q\sqrt[4]{q^{-1}}\cos 2x - q^4(\sqrt[4]{q^{-1}} + 3\sqrt[4]{q^{-9}})\cos 4x - \cdot$$

On the other hand if we write

$$\frac{H^2(z)\Theta_1(z)}{\Theta^2(z)} = \frac{H(z)\Theta_1(z)}{\Theta(z)}\cdot\frac{H(z)}{\Theta(z)},$$

we have

$$\sqrt{\frac{K}{2\pi}\frac{H(z)\Theta_1(z)}{\Theta(z)}} = \sqrt{q}\sin x + \sqrt[4]{q^9}(1 + 2q^{-1})\sin 3x + \cdots,$$

$$\sqrt{\frac{kK}{\pi}}\frac{H(z)}{\Theta(z)} = \frac{2\sqrt{q}}{1-q}\sin x + \frac{2\sqrt{q^3}}{1-q^3}\sin 3x + \cdots.$$

If we multiply these two series, we get another expression for the left side of (5). Comparison of these two developments gives

$$A = \sum F(N)q^{\frac{1}{4}N}, \quad N \equiv 3 \mod 4,$$

* In our review of vol. 1 we noticed, l. c., p. 185, another demonstration by Hermite of this theorem. Cf. Oeuvres, vol. 1, p. 260.

where $F(N)$ is the number of solutions of

$$N = (2n+1)(2n+4b+3) - 4a^2 \quad (a = 0, \quad \pm 1, \cdots \pm n)$$

and n, b are positive integers. On the other hand Hermite shows that $F(N)$ is nothing but Kronecker's function F defined above. Let us now set $x = 0$ in (5). The left side vanishes, and if we arrange the right side according to powers of q, Hermite finds, letting d', d'' be divisors of N such that $d' > \sqrt{N}$, and $d'' < \sqrt{N}$, that

$$A\Theta(0) = \tfrac{1}{2}\Sigma q^{\frac{1}{4}N}(\Sigma d' - \Sigma d'').$$

The coefficient of $q^{\frac{1}{4}N}$ on the right Kronecker calls $\Psi(N)$; the left side we see is the product of two infinite series in q. Performing the multiplication and equating coefficients of like powers of q gives finally

$$F(N) + 2F(N-2^2) + 2F(N-4^2) + \cdots + 2F(N-4k^2) = \tfrac{1}{2}\Psi(N),$$

a relation between the number of properly primitive quadratic forms with the determinants $-N, -(N-4), -(N-16), \cdots$

If we have gone into some details in speaking of the papers 6), 12), and 15), it is partly because their importance demands more than a passing notice and partly with the hope that our remarks may awaken the interest of some reader of this BULLETIN to look farther into these matters.

JAMES PIERPONT.

SHORTER NOTICES.

Serret's Lehrbuch der Differential- und Integralrechnung. Dritte Auflage, dritter Band,* neu bearbeitet von GEORG SCHEFFERS. Leipzig, Teubner, 1909. xii + 658 pp.

THIS book on differential equations is the third and last volume of Scheffer's "Umarbeitung" of the second edition of Serret's Lehrbuch. In comparison with the first two volumes, there are many more alterations made in this third edition of the third volume. In fact one can hardly recognize any traces

*The first two volumes of this work were reviewed in the BULLETIN, vol. 15 (1908–09), p. 140.

of the old book in the new. The arrangement of material has been changed, some subjects have been omitted entirely, some have been developed from mere paragraphs into chapters, the real variable has been separated from the complex, the figures have been redrawn, and although the new edition contains fewer general topics than the old yet it is nearly one half larger. The worked out illustrative examples are left practically unchanged, but many of them have been shifted to more fitting sections.

The first chapter is a thirty-page general discussion of differential equations, ordinary, total, and partial, with a clear statement of the two directions which a course in differential equations might take:

I. "Die Bestimmung aller Lösungen bzw. Lösungensysteme soweit wie mittels Elimination, Substitution und Differentiation möglich auf blosze Quadraturen zurückzuführen."

II. "Unmittelbar aus den vorgelegten Gleichungen oder Systemen von Gleichungen die Eigenschaften der Lösungensysteme zu erkennen."

A broad interpretation of the first point of view dominates the book, but the second is not neglected, about a quarter of the volume being devoted to the function theory side of the subject.

The second chapter consists of the derivation of existence theorems for ordinary differential equations, systems of ordinary differential equations, and implicit functions of one or more variables. This chapter is parenthetical in character and "upon first reading" may be omitted. The real variable alone is considered. The third, fourth, and fifth chapters treat of ordinary differential equations of the first order, systems of ordinary differential equations of the first order, and ordinary differential equations of higher orders respectively. The rather brief discussion of Lie's integration methods based upon infinitesimal transformations which appeared in the second edition has been greatly expanded in these chapters and has been made one of the features of the book, notwithstanding the detailed treatment of the older methods. The sixth chapter, also parenthetical in character, extends the theory of the second chapter into the domain of the complex variable.

Partial differential equations are taken up in the next two chapters. The seventh treats of the linear partial differential equation of the first order, first reducing the problem to that of the homogeneous partial differential equation. A few pages at the end are devoted to Pfaff's equation in three variables. The

eighth chapter takes up the general partial differential equation of the first order by two methods. First the emphasis is laid on the geometrical methods of Lagrange and Monge, which are made to serve as an introduction to the analytical method of Cauchy given in the second part. Partial differential equations of higher orders are not discussed in this edition.

Zermelo's chapter on the calculus of variations in the second edition has been replaced by a shorter chapter with practically the same content. The notation has been changed to conform to the notation of the more recent articles and treatises. There is no treatment of sufficient conditions, merely a discussion of Euler's equations for the simplest problem of the calculus of variations, with some extensions to isoperimetric problems and problems in three variables. The subject matter is illustrated by the usual examples, the catenary, brachistochrone, etc. We note that in this edition, "Die Eulersche Differentialgleichung" replaces "Die Lagrangesche Differentialgleichung," a result probably of Professor Bolza's championship of Euler's claim to priority.

Harnack's appendix on the integration of partial differential equations and the few pages of "Bemerkungen" have been omitted in this edition.

As a third volume in a course in calculus, intended for students in their first three semesters, the present volume will be found rather advanced, notwithstanding the footnotes pointing out paragraphs and chapters which may be omitted. Scientifically, however, the new edition is a vast improvement over the old. The arrangement of the subject matter, the clearness of the language, the precise statement of definition and theorem, the copious index, and the typography place this work among the best reference text-books on differential equations in German or English.

A. R. CRATHORNE.

Kreis und Kugel in senkrechter Projektion, für den Unterricht und zum Selbststudium. Von Dr. OTTO RICHTER. Leipzig und Berlin, Teubner, 1908. x + 187 pp., with 147 figures.

THE author's aim as set forth in the preface is to furnish a supplement to the numerous elementary treatises on descriptive geometry. He proposes to give general solutions of certain fundamental problems which are studied for special cases only in books on descriptive geometry, and to give the student a

knowledge of the principles of stereometry which he has a right to know. The hope is expressed that the book may serve to promote skill in draughting, to increase the reader's ability to see things in space, and to arouse and develop in the youthful mind a sense of accuracy and beauty.

The first chapter is devoted to the ellipse, beginning in a most elementary fashion with the ellipse as an oblique section of a right circular cylinder. The geometrical properties of the curve are developed in detail, as well as all construction problems involved. This chapter may seem unnecessarily drawn out, but the author justifies it on the ground that the beginner with little or no knowledge of conic sections is enabled thus to collect his tools and become acquainted with much which will be of value in subsequent work.

The second chapter treats the sphere in like detail. This chapter introduces the idea of a non-euclidean (elliptic) geometry on a sphere with arcs of great circles as "straight lines," the object then being to represent on a plane the configurations of this geometry. Thus, in analogy to plane geometry, we erect the perpendicular to a straight line at a given point, bisect lines, bisect angles formed by two straight lines, extend two segments of straight lines and find their point of intersection and draw "tangents" to circles. Such problems as the construction of the poles of a plane and a general discussion of rectangular axonometry are included in this chapter.

The third and last chapter is devoted to applications, and forms more than one half of the text. This chapter is less detailed and more suggestive than the preceding ones. Many problems are left for the reader to complete or investigate entirely. The chapter is divided into five sections. The first treats of prisms, pyramids, cylinders, cones, and spheres. Two classes of problems are discussed, those involving relative position only, as inscribing a pyramid in a sphere; and those involving relative magnitudes, as drawing the beam of greatest strength in a given cylindrical log, the section to be rectangular. The drawing of the intersections of these solids is eventually included, but the cones and cylinders are only of the second degree. The second section is devoted to spherical wedges and allied forms. The third includes a classification and general method for representing Archimedean and Platonic bodies, and a "free" drawing of regular polyhedra including the Kepler and Poinsot star polyhedra. The fourth deals with bodies of

revolution and spiral forms, a greater part of the space being devoted to the torus and the spiral stairway. The closing section is concerned with the problem of representing cylinders, cones, and spheres having an orthogonal network of lines on their surface. This leads at once to drawing maps of the earth's surface and of the celestial regions. Throughout the last chapter numerous and really wonderful examples of regular and semiregular forms occurring in nature are pointed out, and in a few cases the principles of the text are employed in drawing a flower or plant.

While a portion of the book could be read with advantage by the beginner, it would seem to the reviewer to be most useful to the teacher in suggesting methods and examples. It is much more mathematical than American text-books on descriptive geometry, and less extensive than many of the German books on that subject, such as Geyger. On the other hand it does not cover the field of stereometry in general on such an ambitious scale as does Dr. Holzmüller in his four-volume Elemente der Stereometrie. In many cases the figures are rendered unnecessarily complicated by the attempt to use the same figure for a number of examples, often separated by many pages. The author has undoubtedly succeeded in thoroughly discussing the limited field he selects, and the reader who covers the book carefully will be certain to gratify the author's desire to increase his power of space perception.

D. D. LEIB.

Plane and Spherical Trigonometry and Four-Place Tables of Logarithms. By W. A. GRANVILLE. Boston, Ginn and Company, 1909. xi + 264 + 38 pp.

THIS is one of the excellent series of elementary mathematical text-books which are published under the supervision of Professor P. F. Smith, of the Sheffield Scientific School. The present volume is up to the standard for which the series has already earned a reputation.

The number of text-books in trigonometry is growing larger every day, but there always seems to be room for one more, provided it is written in a modern spirit so as to satisfy the needs of the present day. This book covers the usual topics and contains all the trigonometry that is usually taught in the undergraduate classes of colleges and technical schools. The demonstrations are simple and exceedingly clear, and the book

is abundantly rich in exercises. With regard to the numerical work it is evident that the author believes in the pedagogic value of accuracy and the proper arrangement of computation. Great prominence has also been given to graphical methods.

The book differs from the general type of text-books in trigonometry in several points. Under each case in the solution of triangles there are two sets of examples, one in which the angles are given in degrees and minutes, and another in which they are expressed in degrees and the decimal part of a degree. Another feature is the treatment of spherical trigonometry, in deriving the formulas of which the author makes use of the principle of duality, which is stated in the following form: "If the sides of a general spherical triangle are denoted by the Roman letters a, b, c, and the supplements of the corresponding opposite angles by the Greek letters, α, β, γ, then, from any given formula involving any of these six parts, we may write down a dual formula by simply interchanging the corresponding Greek and Roman letters." This method has many advantages, a great part of the work required in deriving formulas being done away with.

The general appearance of the book is very attractive. The cuts, the typography, and the arrangement of matter on the page are excellent.

JACOB WESTLUND.

Dynamique Appliquée. Par L. LECORNU. Paris, Octave Doin, 1908. 534 pp. 5 francs.

Hydraulique Générale. Par A. BOULANGER. Paris, Octave Doin, 1908. Vol. 1, xvi + 382 pp. Vol. 2, vii + 299 pp. 5 francs each.

(Encyclopédie Scientifique. Publiée sous la direction du Dr. Toulouse.)

M. LECORNU's textbook on applied mechanics is divided into four parts: Résumé of the chief results of rational mechanics; Mechanical properties of materials; Applications of dynamics; Theory of machines.

The first part occupies seventy-one pages, and furnishes the theoretical foundation for the rest of the book. It is a summary of the well-known equations of translations, moments; kinematics of a point, a set of points, and a solid; dynamics of particles, momentum, lifting force; statics of systems, virtual work, restraints, equilibrium of solids and of jointed systems,

funiculars, statics of fluids; dynamics of free systems; dynamics of restrained systems; dynamics of rigid bodies; dynamics of fluids. The subject is evidently sufficiently vast, and one could scarcely expect to find in the small space assigned even a complete résumé. The author has succeeded however in including the chief results of a course in theoretical mechanics, rather as a review to the student than as a formulary for reference.

In the second part three topics are considered: friction, which receives very careful attention, resistance of the air, and impact. Under friction we find discussed the friction of journals, gears, screws, the universal joint, sliding pieces, friction clutches, skidding, cord on cylinder, cylinder on inclined plane, billiard ball, the tendency to extinguish the friction. These topics give some idea of the completeness of the discussion. The next chapters treat rolling friction, pivotal friction, and the stiffness of ropes.

The third part considers applications of theoretical dynamics to springs, to Watt's indicator, and to pendular movements — these rather fully. More briefly, the top, the hoop, the bicycle, the balancing of wheels, the equilibration of turbines. The fourth part discusses machines: in particular, motors, governors, brakes, and transmission of power.

The object of the book is to exemplify the great practical advantage in attacking problems of applied mechanics directly by the methods of rational mechanics. "It has for principal object the study of the general properties of machines, excepting questions concerning thermodynamics or electricity." The author has succeeded in his object and the illustrations he uses are very well chosen. It would be well if some such exemplification as this could be introduced into our textbooks on theoretical mechanics.

The latter of the two books under review is avowedly a systematic résumé of the results and methods of Professor J. Boussinesq, first presented in various memoirs treating of the movements of water. The problems discussed are well classified and treated with much completeness. The introduction establishes the fundamental laws of hydrodynamics; the first section discusses phenomena in which friction is negligible; the second section, phenomena in which friction has a sensible influence; the third section, the phenomena of turbulent movements. In

the second volume, which is devoted to singularities and applications, there are also three sections with the same headings as in the first volume.

The swell on the ocean is first discussed, followed by the solitary wave or wave of translation, and waves of emersion and impulsion. The flow of water through fine tubes, the phenomena of filtration, and rotatory flow are taken up in the next section. Under turbulent movement we find the uniform and the gradually varied flow in conduits and canals.

In the second volume the topics are: flow through orifices, flow over weirs, the extinction of the ocean swell and waves, resistance to immersed solids, motion in conduits and canals with singularities such as enlargements, contractions, bends, etc., natural water courses, motion in rectangular canals with eddies, waves in elastic conduits, and the water hammer.

The treatment of the subject is deep enough for the usual student and tends to incite reading of the original memoirs of investigators in this field. It is also calculated to interest the student of physics on the one hand and the scientific engineer on the other. If such courses as this were introduced into the senior year of our engineering colleges in place of certain customary "practical" courses which could just as well be read up outside by the intelligent student (and no other should be considered!), it would tend to improve the scientific study of many engineering problems now worked out only empirically and approximately. Particularly as the laws developed here are in general experimentally verifiable. In fact the title might be translated into Theoretical Hydraulics in contradistinction to Practical Hydraulics. With the definiteness found in the treatment throughout, the engineering student can scarcely fail to look upon his problems from a much higher point of view than if he has only the usual course in hydraulics.

These two treatises exemplify very well the scientific character of the Encyclopédie Scientifique. If all the projected volumes are kept up to the same level, the whole will undoubtedly be a collection which every student should possess. At least it would serve in America as a corrective to too much practicalism, and would encourage the scientific spirit in engineering. It is needless to say that courses in "engineering mathematics" would not suffice for the reading of such works as these.

JAMES BYRNIE SHAW.

Die Theorie der Besselschen Funktionen. Von PAUL SCHAFHEITLIN. Leipzig, Teubner, 1908. v + 129 pp.

B. G. TEUBNER is publishing a series of mathematical and physical monographs for engineers. Each volume treats some topic in mathematics or in physics and in a hundred pages gives the busy engineer a general idea of the subject and a short account of its history, proves some theorems and gives others unproved, tells him something of the applications and gives him references to more complete works.

The volume under review is number four of the series. It is designed to give a reader who knows only the elements of the calculus a working knowledge of the theory of Bessel's functions. The introduction gives a short history of the functions, a list of important articles and treatises, and mentions the applications to hydrodynamics, wireless telegraphy, vibrations of a membrane, etc. Then comes a collection of some thirty unproved theorems and formulas, chiefly in hypergeometric series and gamma functions, to which the author refers in the course of the book. The subject matter proper begins with a solution of Bessel's differential equation in series by undetermined coefficients, and a definition of Bessel's function of the first kind. Since no knowledge of differential equations is assumed on the part of the reader, a short discussion of linear homogeneous differential equations and their solutions is given. Then follow with proofs the usual formulas giving the relations between Bessel's functions of different orders and their derivatives and the different expressions for the Bessel's functions of the first kind. In particular the case in which the parameter is an integer and the case in which it is the half of an odd number are discussed. The remaining chapters take up Bessel's functions of the second kind, the representation of arbitrary functions by Bessel's functions, the addition and multiplication theorems and the variation, graphical representation, and roots of the functions for various orders. Throughout the book important theorems and formulas are numbered in heavy type and these collected together form an appendix to the volume. The book is a mine of information about Bessel's functions, but the engineer will wish for a fuller discussion of the applications.

A. R. CRATHORNE.

CORRECTION.

In Miss Noether's abstract on page 116 of the January BULLETIN the last sentence should read: The two fundamental theorems of the symbolic method, that concerning invariant processes and that regarding the totality of the invariant identities, readily appear by this method.

In the February BULLETIN, page 274, the amount of the prize granted by the Göttingen academy to Dr. Wieferich should have been stated as 1,000 Marks. Dr. Wieferich's paper appeared in volume 136 of *Crelle*.

NOTES.

THE twenty-sixth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on Friday and Saturday, April 8, 9, 1910, the first session opening at 10 o'clock a. m. in Room 32, Ryerson Physical Laboratory.

THE lectures delivered before the New Haven Colloquium of the American Mathematical Society in September, 1906, by Professors E. H. MOORE, E. J. WILCZYNSKI, and MAX MASON are now in press and will soon be issued in book form. The lectures are published by Yale University.

AT the meeting of the London mathematical society held on February 10, the following papers were read: By H. W. RICHMOND, "A note on double series of lines"; by H. LAMB, "On the diffraction of a solitary wave"; by H. F. BAKER, "Notes on various points in the theory of functions."

THE valuable library on mathematics and science of the late OREN ROOT, for many years professor of mathematics at Hamilton College, has been presented to the college by his son, Senator ELIHU ROOT.

UNDER the auspices of the Italian ministry of public instruction and the editorial direction of Professors V. Volterra, G. Loria and D. Gambioli the first complete edition of the mathematical works of Count JULIUS CHARLES DI FAGNANO will be

published in the course of the year. The edition will include his papers of a controversial character and scientific correspondence, several valuable documents of which have been recently discovered in the Oliveriana library in Pesaro. In order to make this collection as complete as possible, the editors request the coöperation of any persons who can supply information as to the existence in public or private libraries of manuscripts regarding Count Fagnano.

THE mathematical papers of the late Professor H. MINKOWSKI are being collected and edited by Professor D. HILBERT. They will be published in two octavo volumes by Teubner during the summer, and will include all his writings except the two books, Geometrie der Zahlen and Diophantische Approximationen. The memoirs are arranged in four groups, comprising the theory of numbers, the geometry of numbers, geometry, and physics. An extensive manuscript concerning the theory of convex bodies will appear for the first time in the forthcoming publication.

THE firm of Macmillan and Company of London announce the publication of a two-volume biography of Sir William Thomson by Professor S. P. THOMPSON. While the present work also considers his early life, it does not encroach upon the family narrative which has just been edited by Lord Kelvin's nieces.

THE prize of 1,200 francs announced by the Istituto Lombardo (at Milan) for an important contribution to the theory of continuous groups has been awarded to Professor U. AMALDI, of the University of Modena, for a paper entitled "I gruppi continui infiniti di transformazioni puntuali della spazio a tre dimensioni."

THE next summer meeting of the British association for the advancement of science will be held at Sheffield beginning August 31, under the presidency of Professor T. G. BONNEY. Professor E. W. HOBSON is chairman of section A, mathematics and physics.

DURING the interval from April 11 to April 22 a vacation course in mathematics for teachers of secondary schools will be held at the University of Göttingen. The following programme is announced: By Professor H. BEHRENDSEN: Geo-

metric methods in instruction in arithmetic, four hours.—By Professor F. KLEIN: Elementary mechanics, six hours.—By Professor L. PRANDTL: Aerial dynamics and ballooning, four hours.—By Professor E. LANDAU: Theory of aggregates, four hours.

THE following university courses are announced for the summer semester:

UNIVERSITY OF BONN.—By Professor E. STUDY: Theory of elliptic functions, four hours; Seminar, two hours.—By Professor F. LONDON: Descriptive geometry with exercises, four hours; Selected chapters of analytic geometry, two hours: Seminar, two hours.—By Professor S. HESSENBERG: Foundations of geometry, two hours; Seminar, two hours.—By Dr. J. O. MÜLLER: Theory of numbers, three hours.—By ————: Differential and integral calculus, four hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Analytic geometry of the plane, four hours; Selected portions of the theory of functions, two hours; Seminar.—By Professor H. GRASSMANN: Ordinary differential equations, four hours; Descriptive geometry, II, five hours; Seminar.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Applications of the calculus to geometry, four hours; Seminar on the boundary between mathematics and philosophy (with Professor Zermelo), two hours.—By Professor D. HILBERT: Principles of mathematics, four hours; Selected chapters in the theory of partial differential equations, two hours; Seminar, two hours.—By Professor E. LANDAU: Differential and integral calculus, with exercises, five hours; Selected chapters in the theory of functions, two hours; Concerning the Fermat theorem, one hour; Seminar, two hours.—By Professor C. RUNGE: Analytic geometry with exercises, four hours; Seminar, two hours.—By Professor L. PRANDTL: Introduction to thermodynamics, three hours; Seminar, two hours.—By Professor E. ZERMELO: Differential equations, three hours; Seminar (with Dr. Toeplitz and Dr. Born), two hours.—By Dr. P. KOEBE: Introduction to synthetic geometry, two hours; Riemann's theory of functions, four hours.—By Dr. O. TOEPLITZ: Introduction to the theory of integral equations, and of an infinite number of variables, four hours.—By Dr. F. BERNSTEIN: Quadratic and relative quadratic number fields, two hours;

Calculation of insurance, one hour.—By Dr. C. H. MÜLLER: Mathematics of Archimedes, one hour; Proseminar in mechanics of rigid bodies, two hours.

UNIVERSITY OF GREIFSWALD. — By Professor W. THOMÉ: Theory of analytic and particularly elliptic functions I, four hours; Algebraic surfaces and space curves, two hours; Seminar. — By Professor F. ENGEL: Differential and integral calculus I, four hours; Analytic geometry of space, two hours; Seminar. — By Professor K. T. VAHLEN: Differential geometry, three hours, with exercises, one hour.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Integral calculus, five hours; Theory of substitutions and higher algebraic equations, four hours; Seminar, two hours. — By Professor A. VOSS: Analytic geometry of space, four hours; Calculus of variations, three hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Selected chapters in the theory of analytic functions, four hours. — By Professor A. SOMMERFELD: Partial differential equations of physics for beginners, three hours; with exercises, one hour. — By Professor H. BRUNN: Elements of higher mathematics for students of all faculties, four hours. — By Professor K. DOEHLEMANN: Descriptive geometry, II, four hours; with exercises, 2 hours; Synthetic geometry, II, two hours; with exercises, two hours. By Dr. H. HARTOGS: Theory of abelian functions, II, with applications to the theory of algebraic curves, three hours. — By Dr. O. PERRON: Selected chapters of elementary geometry, five hours; Theory of linear differential equations, four hours. By Dr. R. WAGNER: Mathematical supplement to the lectures on experimental physics, one hour.

PROFESSOR E. ZERMELO, of Göttingen, has been appointed professor of mathematics at the University of Zurich.

DR. A. BUHL has been appointed professor of general mathematics at the University of Toulouse.

PROFESSOR A. R. FORSYTH, of the University of Cambridge, has resigned the Sadlerian professorship of mathematics. The vacancy has been filled by the election of Dr. E. W. HOBSON, of Christ's College.

AT the University of Berlin, Professor G. FROBENIUS and Professor F. SCHOTTKY have been given the title of Geheimer Regierungsrat.

DR. —. HELLEBRAND, of the agricultural institute of Vienna, has been promoted to an associate professorship of mathematics.

THE mathematical staff of Columbia University has been enlarged by the appointment of Professor W. B. FITE, of Cornell University, and H. E. HAWKES, of Yale University, as professors of mathematics.

MR. H. H. MITCHELL, of Princeton University, has been appointed instructor in mathematics in the Sheffield Scientific School of Yale University.

PROFESSOR A. CAPELLI, of the University of Naples, died January 28 at the age of 55 years. He was a member of the Italian Society of Sciences, of the Accademia dei Lincei, etc.

PROFESSOR F. PURSER, of the University of Dublin, author of several memoirs on the Bessel functions, died January 28 at the age of 70 years.

BOOK catalogues: A. Hermann et Fils, 6 rue de la Sorbonne, Paris, catalogue for 1910, about 150 titles in mathematics. — H. Sotheran and Company, 140 Strand, London, price current No. 702, about 200 titles in mathematics.

NEW PUBLICATIONS.

(*In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor,* PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.*)

I. HIGHER MATHEMATICS.

BACHMANN (P.). Niedere Zahlentheorie. Zweiter Teil: Additive Zahlentheorie. Leipzig, Teubner, 1910. 8vo. 10 + 480 pp. Cloth. M. 17.

BAUER (G.). Vorlesungen über Algebra. 2te Auflage, herausgegeben vom Mathematischen Verein München. Leipzig, Teubner, 1910. 8vo. 6 + 366 pp. Cloth. M. 12.

DANIELSEN (O.). Nogle bemærkninger om en gruppe algebraiske flader, der kunne bringes til at svare entydig til en plan punkt for punkt. Kjöbnhavn, 1909. 8vo. 92 pp. M. 3.60

FABRY (E.). Problèmes et exercices de mathématiques générales. Paris, Hermann, 1900. 8vo. 424 pp. Fr. 10.00

FRÉCHET. See HADAMARD (J.).

FREUND (E.). Entwicklung willkürlicher Funktionen vermittelst meromorpher. (Diss.) Breslau, 1909. 8vo. 60 pp.

GEUS (A.). Die eindeutigen Transformationen der ebenen Kurve dritter Ordnung in sich, invarianten- und funktionentheoretisch behandelt. (Diss.) Erlangen, 1909. 8vo. 43 pp.

HADAMARD (J.). Leçons sur le calcul des variations. Recueillies par M. Fréchet. Vol. I: La variation première et les conditions du premier ordre; les conditions de l'extremum libre. Paris, Hermann, 1910. 8vo. 8 + 520 pp. Fr. 18.00

HÖEGH (E. VON). Elementarer Beweis des Fermatschen Satzes. Rostock, 1909. M. 1.50

IHLENBURG (W.). Ueber die geometrischen Eigenschaften der Kreisbogenvierecke. Leipzig, 1909. M. 8.50

KOWALEWSKI (G.). Die klassischen Probleme der Analysis des Unendlichen. Ein Lehr- und Uebungsbuch für Studierende zur Einführung in die Infinitesimalrechnung. Leipzig, Engelmann, 1910. 8vo. 8 + 383 pp. Cloth. M. 16.50

LACHELIER (H.). See LEIBNIZ.

LA COUR (P.). Historisk Matematik. 3. udgave. III: Bogstavregning. Kjöbnhavn, 1909. 8vo. 72 pp. M. 1.50

LEHMER (D. N.). Factor table for the first ten millions, containing the smallest factor of every number not divisible by 2, 3, 5, or 7 between the limits 0 and 10,017,000. Washington, Carnegie Institution, 1909. Folio. 476 pp.

LEIBNIZ. La monadologie. Publiée d'après les manuscrits de la bibliothèque de Hanovre, avec introduction, notes, et suppléments, par H. Lachelier. 7e tirage. Paris, Hachette, 1909. 16mo. 103 pp. Fr. 1.00

MANNING (H. P.). The fourth dimension simply explained. A collection of popular essays, with an introduction and editorial notes by Henry P. Manning. New York, Munn, 1910. 8vo. 251 pp. Cloth. $1.50

MEYER (W. F.). Allgemeine Formen- und Invariantentheorie. Vol. I: Binäre Formen. Leipzig, 1909. M. 9.60

NIEWIADOMSKI (R.). Analyse de l'équation $z^n = x^n + y^n$, avec une résolution du grand théorème de Fermat. Varsovie, 1909. 8vo. 40 pp. M. 2.50

SCHOY (C.). Beiträge zur konstruktiven Lösung sphärisch-astronomischer Aufgaben. Leipzig, Teubner, 1910. 8vo. 7 + 40 pp. M. 1.60

SYLVESTER (J. J.). Collected mathematical papers. Vol. 3. Cambridge, University Press, 1909. 8vo. Cloth. 18s.

ZIEMKE (E.). Ueber partielle Differentialgleichungen erster Ordnung mit Integralvereinen, die als Punktmannigfaltigkeiten zweifach ausgedehnt sind. (Diss.) Greifswald, 1909. 8vo. 73 pp.

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

BELLENGER (H.). See NEVEU (H.).

BÖGER (R.). Elemente der Geometrie der Lage, für den Schulunterricht bearbeitet. 2te Auflage. Leipzig, Göschen, 1910. 16mo. 45 pp. M. 0.90

—— Projektive und analytische Schulgeometrie. Ein Lehr- und Uebungsbuch für die Oberklassen. Leipzig, Göschen, 1910. 8vo. 8 + 211 pp. Cloth. M. 3.60

BONET Y GARCÍA (J.). Apuntes de algebra elemental. 3a edición corregida y aumentada. Cuaderno 1°: Pizarras. Cuaderno 2°: Aclaraciones. Madrid, 1909. 2 vols., 104 and 137 pp. P. 8.50

BRIOT (C.) et VACQUANT (C.). Elementos de geometría aplicáda. Paris, Hachette, 1909. 16mo. 436 pp. Fr. 3.50

BÜTZBERGER (F.). Lehrbuch der ebenen Trigonometrie. Resultate der Aufgaben. Zürich, 1910. 8vo. 35 pp. M. 2.00

CAMINATI (P.). La somma degli angoli d'ogni triangolo rettilineo è sempre precisamenie equivalente alla somma di due angoli retti: dimostrazione elementare indipendente dal V postulato e dall' XI assioma di Euclide. Parma, Zerbini, 1909. 4to. 4 pp.

CANNAVIELLO (M.). Corso die geometria elementare, per le scuole medie. Parte II: Stereometria. Napoli, Simeone, 1909. 8vo. Pp. 323–469. L. 1.30

COHN (B.). Tafeln der Additions- und Subtraktions-Logarithmen auf sechs Dezimalen. Leipzig, 1909. M. 4.00

COMBET (E.). Tables abrégées de logarithmes, de lignes trigonométriques naturelles, de nombres usuels et de constantes physiques. Paris, Belin, 1909. 16mo. 15 pp. Fr. 0.50

ENRIQUES (F.) et AMALDI (U.). Elementi di geometria, ad uso delle scuole tecniche. Bologna, Zanichelli, 1909. 16mo. 7 + 254 pp. L. 2.00

GODFREY (C.) and SIDDONS (A. W.). Elementary geometry. Part 2, theoretical. Cambridge, University Press, 1910. 8vo. 344 pp. Cloth. 3s.

GRÉVY (A.). Géométrie à l'usage des élèves des classes de cinquième B, troisième B (programmes du 31 mai 1902). 7e édition. Paris, Vuibert, 1910. 16mo. 323 pp.

INSKIP (G. D.). A new manual of squares and logarithms from 0 feet to 100 feet, advancing by 32nds of inches. Logarithms of numbers from 0 to 10000, logarithmic trigonometric functions for each minute and the auxiliary trigonometric functions S and T for the calculation of the logarithms of sines, tangents and cotangents. London, Wesley, 1910. 8vo. 546 pp. Cloth. 12s.

INTERMEDIATE applied mathematics papers. Questions set at the University of London from 1892 to 1909. London, Clive, 1909. 8vo. 88 pp. 2s. 6d.

KAMBLY und ROEDER. Trigonometrische und stereometrische Lehraufgabe der Untersekunda. Sonderdruck aus der 148–151ten Auflage der Planimetrie, neu bearbeitet von A. Thaer. Breslau, Hirt, 1909. 8vo. Pp. 177–240. M. 0.80

MAHLERT (A.). See MÜLLER (H.).

MARTÍN MENGOD (A.). Elementos de geometría, y trigonometría rectilínea y esférica. Málaga, La Española, 1909. 161 pp. P. 4.00

—— Programa de aritmética y algebra. Málaga, La Española, 1909. 20 pp. P. 1.00

MÉNDEZ SORET (L.). Programa de nociones y ejercicios de geometría. Málaga, La Española, 1909. 15 pp. P. 1.00

MÜLLER (H.) und MAHLERT (A.). Lehr- und Uebungsbuch der Arithmetik und Algebra für Studienanstalten. Ausgabe A: Für gymnasiale Kurse. 1ter Teil: Bis zur Lehraufgabe der Klasse IV. Leipzig, Teubner, 1909. 8vo. 6+181 pp M. 2.40

NEVEU (H.) et BELLENGER (H.). Cours de géométrie théorique et pratique à l'usage des élèves des écoles primaires supérieures et des candidats aux Ecoles d'arts et métiers. 1re partie. Géométrie plane. 2e édition. Paris, Masson, 1909. 16mo. 470 pp. Fr. 3.50

PAHL (F.). See SCHULZE (F.).

PETERS (J.). New calculating tables for multiplication and division by all numbers from 1 to 10,000. Berlin, Reimer, 1909. 6+500 pp. Cloth. M. 15.00

POTTER (W. J.). Concurrent, practical and theoretical geometry. Parts 1–3. London, Holland, 1910. 8vo. 704 pp. 4s. 6d.

ROEDER. See KAMBLY.

SAINT-PAUL (H. DE). Tables des lignes trigonométriques naturelles des angles et des arcs, variant de minute en minute, depuis 0° jusqu'à 90°. Paris, Gauthier-Villars, 1910. 8vo. 32 pp. Fr. 1.50

SALOMON (MME. A.). Leçons d'algèbre à l'usage de l'enseignement secondaire des jeunes filles (classes de quatrième et cinquième années) et des aspirantes au brevet supérieur. Paris, Vuibert, 1910. 16mo. 6+229 pp. Fr. 2.00

SCHULZE (E.) und PAHL (F.). Mathematische Aufgaben. Ausgabe für Realgymnasien und Oberrealschulen. Ergebnisse. 2ter Teil. Leipzig, Dürr, 1909. 8vo. 112 pp. M. 5.00

SCOTTI (G.). Elementi di geometria intuitiva, ad uso del ginnasio inferiore e dei corsi complementari. Tredicesima edizione. Torino, Salesiana, 1910. 8vo. 139 pp. L. 1.00

SIDDONS (A. W.). See GODFREY (C.).

THAER (A.). See KAMBLY.

TUÑÓN DE LARA (M.). Programa de geometría. Madrid, Ducazcal, 1910. 32 pp. P. 1.00

VACQUANT (C.). See BRIOT (C.).

III. APPLIED MATHEMATICS.

BIRCK (O.). See SCHWARZSCHILD (K.).

BOREL (E.). See VESSILLIER (G.).

BOWLEY (A. L.). An elementary manual of statistics. London, Macdonald, 1910. 8vo. 224 pp. Cloth. 5s.

COOPS (G. H.). Uebersichtliche Darstellung des 2ten Hauptsatzes der Thermodynamik und der daraus herzuleitenden Folgen. Groningen, 1909. M. 0.75

COTTERILL (J. H.) and SLADE (J. H.). Lessons in applied mechanics. Vols. I and II. London, Macmillan, 1910. Each volume. 3s.

DARWIN (G. H.). Scientific papers. Vol. 3. Cambridge, University Press, 1910. 8vo. 15s.

EBERHARDT (C.). Theorie und Berechnung der Luftschrauben. Mit Beispielen und Versuchsresultaten aus der Praxis. Berlin, Krayn, 1910. 8vo. 122 pp. M. 7.50

GEOMETRICAL solutions derived from mechanics; a treatise of Archimedes. Translated by J. L. Heiberg. London, Paul, 1910. 8vo. 2s. 6d.

GRETHER (H.). Ueber Potentialbewegung tropfbarer Flüssigkeiten in gekrümmten Kanälen. (Diss.) Karlsruhe, 1909. 4to. 118 pp.

GUICHARD (C.). Traité de mécanique à l'usage des élèves de mathématiques A et B et des candidats aux écoles. 6e édition. Paris, Vuibert, 1909. 8vo. 7 + 248 pp.

GUILLAUME (C. E.). Initiation à la mécanique. Ouvrage étranger à tout programme, dédié aux amis de l'enfance. 2e édition, revue. Paris, Hachette, 1909. 16mo. 14 + 214 pp. Fr. 2.00

HEIBERG (J. L.). See GEOMETRICAL.

IZART (J.). Mécanique. 32e édition. Paris, Dunod, 1910. 16mo. 451 pp. Fr. 3.50

JULLIOT DE LA MORANDIÈRE (L.). De la réserve mathématique des primes dans l'assurance. (Thèse.) Paris, Larose, 1909. 8vo. 604 pp.

POINCARÉ (H.). See VESSILLIER (G.).

SCHWARZSCHILD (K.) und BIRCK (O.). Tafeln zur astronomischen Ortsbestimmung im Luftballon bei Nacht, sowie zur leichten Bestimmung der mitteleuropäischen Zeit an jedem Orte Deutschlands. Göttingen, 1909. M. 3.80

SLADE (J. H.). See COTTERILL (J. H.).

VESSILLIER (G.). Théorie des systèmes géométriques (à masses égales) appliqués aux chances simples de la roulette. Avec des lettres de H. Poincaré et de E. Borel. Paris, Delarue, 1909. Fr. 20.00